Trigonometry and Matrices

Function of a complex variable

Prove that

$$\log\left(\frac{\sin(x+iy)}{\sin(x-iy)}\right) = 2i\tan^{-1}(\cot x \tanh y)$$

Sol.

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$= \alpha + i\beta$$

$$\log(\sin(x+iy)) = \log(\alpha + i\beta)$$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

$$\log[\sin(x+iy)] = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha} \quad \text{—(i)}$$

Change i to $-i$

$$\log[\sin(x-iy)] = \frac{1}{2}\log(\alpha^2+\beta^2) - i\tan^{-1}\frac{\beta}{\alpha} \quad \text{—(ii)}$$

L.H.S $= \log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]$

$$= \log[\sin(x+iy)] - \log[\sin(x-iy)]$$

$$= \frac{1}{2}\cancel{\log(\alpha^2+\beta^2)} + i\tan^{-1}\frac{\beta}{\alpha} - \frac{1}{2}\cancel{\log(\alpha^2+\beta^2)} + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= 2i\tan^{-1}\frac{\beta}{\alpha}$$

$$= 2i \tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right)$$

$$= 2i \tan^{-1}(\cot x \tanh y) = \underline{\underline{R.H.S}}$$