Trigonometry and Matrices

Function of a complex variable

$\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

Prove that $\phi = \frac{1}{2}\log\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right)$

Sol:

$\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

$\cos\theta\cos(i\phi) - \sin\theta\sin(i\phi) = r\cos\alpha + ir\sin\alpha$

$\cos\theta\cosh\phi - i\sin\theta\sinh\phi = r\cos\alpha + ir\sin\alpha$

equate real and imaginary parts

$$r\cos\alpha = \cos\theta\cosh\phi \quad \text{— (i)}$$

$$r\sin\alpha = -\sin\theta\sinh\phi \quad \text{— (ii)}$$

divide (ii) by (i)

$$\frac{\cancel{r}\sin\alpha}{\cancel{r}\cos\alpha} = \frac{-\sin\theta\sinh\phi}{\cos\theta\cosh\phi}$$

$$-\frac{\sin\alpha}{\cos\alpha}\,\frac{\cos\theta}{\sin\theta} = \tanh\phi$$

$$-\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta} = \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}}$$

By Componendo and dividendo

$$\frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta} = \frac{e^{\phi} + \cancel{e^{-\phi}} + e^{\phi} - \cancel{e^{-\phi}}}{e^{\phi} + e^{-\phi} - (e^{\phi} - e^{-\phi})}$$

$$\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \frac{2e^{\phi}}{\cancel{e^{\phi}} + e^{-\phi} - \cancel{e^{\phi}} + e^{-\phi}}$$

$$\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \frac{\cancel{2}e^{\phi}}{\cancel{2}e^{-\phi}}$$

$$\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = e^{\theta+\theta} = e^{2\theta}$$

Taking log both sides

$$\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right] = \log_e{}^{2\theta}$$

$$\log\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right) = 2\theta \log_e$$

$$\log\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right) = 2\theta$$

$$\phi = \frac{1}{2}\log\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right)$$