

## Trigonometry and Matrices

### Function of a complex variable

$$\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$$

$$\text{Prove that } \phi = \frac{1}{2} \log\left(\frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}\right)$$

Sol.  $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

$$\cos\theta \cos(i\phi) - \sin\theta \sin(i\phi) = r\cos\alpha + ir\sin\alpha$$

$$\cos\theta \cosh\phi - i\sin\theta \sinh\phi = r\cos\alpha + ir\sin\alpha$$

Equate real and imaginary parts

$$x \cos \alpha = \cos \theta \cosh \phi \quad - \textcircled{I}$$

$$x \sin \alpha = -\sin \theta \sinh \phi \quad - \textcircled{II}$$

divide  $\textcircled{II}$  by  $\textcircled{I}$

$$\frac{x \sin \alpha}{x \cos \alpha} = \frac{-\sin \theta \sinh \phi}{\cos \theta \cosh \phi}$$

$$-\frac{\sin \alpha}{\cos \alpha} \frac{\cos \theta}{\sin \theta} = \tanh \phi$$

$$-\frac{\sin \alpha \cos \theta}{\cos \alpha \sin \theta} = \frac{e^\phi - e^{-\phi}}{e^\phi + e^{-\phi}}$$

By Componendo and dividendo

$$\frac{\cos \alpha \sin \theta - \sin \alpha \cos \theta}{\cos \alpha \sin \theta + \sin \alpha \cos \theta} = \frac{e^{\alpha} + \cancel{e^{-\alpha}} + e^{\theta} - \cancel{e^{-\theta}}}{e^{\alpha} + e^{-\alpha} - (e^{\theta} - e^{-\theta})}$$

$$\frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} = \frac{2e^{\theta}}{\cancel{e^{\alpha}} + e^{-\alpha} - \cancel{e^{\theta}} + e^{-\theta}}$$

$$\frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} = \frac{\cancel{\alpha} e^{\theta}}{\cancel{\alpha} e^{-\theta}}$$

$$\frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} = e^{\theta + \theta} = e^{2\theta}$$

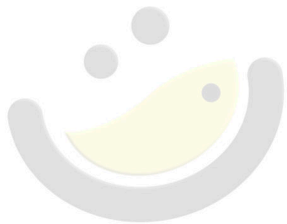
Taking log both sides

$$\log \left[ \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right] = \log e^{2\theta}$$

$$\log \left( \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right) = 2\theta \log e$$

$$\log \left[ \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right] = 2\theta$$

$$\phi = \frac{1}{2} \log \left( \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right)$$



**OMG { MATHS }**  
The poetry of logical ideas.