Trigonometry and Matrices

Function of a complex variable

If $i^{i^{i^{\cdots\infty}}} = A + iB$ and Only principal Values are Considered

Prove that

(i) $\tan \dfrac{\pi A}{2} = \dfrac{B}{A}$

(ii) $A^2 + B^2 = e^{-\pi B}$

Sol.

$$i^{i^{i^{\cdots\infty}}} = A + iB$$

$$i^{A+iB} = A + iB$$

$$A + iB = i^{(A+iB)}$$

$$= e^{(A+iB)\log i}$$

$$= e^{(A+iB)\log(\cos\pi/2 + i\sin\pi/2)}$$

$$= e^{(A+iB)\log_e e^{i\pi/2}}$$

$$a^z = e^{z\log a}$$

$$\left[e^{ix} = \cos x + i\sin x\right]$$

$$= e^{(A+iB)\,i\pi/2}$$

$$= e^{\frac{Ai\pi}{2} + i^2\frac{B\pi}{2}}$$

$$= e^{-\frac{B\pi}{2} + \frac{A\pi}{2}i}$$

$$= e^{-\frac{B\pi}{2}} \cdot e^{\frac{A\pi}{2}i}$$

$$= e^{-\frac{B\pi}{2}} \left[\cos\frac{A\pi}{2} + i\sin\frac{A\pi}{2}\right]$$

$$\left(e^{ix} = \cos x + i\sin x\right)$$

$$A + iB = e^{-\frac{B\pi}{2}} \cos\frac{A\pi}{2} + i e^{-\frac{B\pi}{2}} \sin\frac{A\pi}{2}$$

Compare real and imaginary parts

$$A = e^{-\frac{B\pi}{2}} \cos\frac{A\pi}{2} \quad \text{—(I)}$$

$$B = e^{-B\pi/2} \sin\frac{A\pi}{2} \quad \text{—(II)}$$

Divide (II) By (I)

$$\frac{B}{A} = \frac{\cancel{e^{-B\pi/2}} \sin A\pi/2}{\cancel{e^{-B\pi/2}} \cos A\pi/2}$$

$$\frac{B}{A} = \tan \frac{A\pi}{2}$$

Sq. are adding (I) & (II)

$$A^2 + B^2 = e^{-B\pi} \cos^2 \frac{A\pi}{2} + e^{-B\pi} \sin^2 \frac{A\pi}{2}$$

$$= e^{-B\pi}\left(\cos^2\frac{A\pi}{2} + \sin^2\frac{A\pi}{2}\right)$$

$$A^2 + B^2 = e^{-B\pi}$$