Trigonometry and Matrices

Function of a complex variable

$$\frac{(1+i)^{p+iq}}{(1-i)^{p-iq}} = \alpha + i\beta \quad \text{Prove that}$$

$$\tan^{-1}\frac{\beta}{\alpha} = \frac{p\pi}{2} + q\log 2.$$

Sol.

$$\frac{(1+i)^{p+iq}}{(1-i)^{p-iq}} = \frac{e^{(p+iq)\log(1+i)}}{e^{(p-iq)\log(1-i)}}$$

$$a^z = e^{z\log a}$$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

$$= \frac{e^{(p+iq)\left[\frac{1}{2}\log(2) + i\tan^{-1}1\right]}}{e^{(p-iq)\left[\frac{1}{2}\log 2 - i\tan^{-1}1\right]}}$$

$$= \frac{e^{(p+iq)\left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right]}}{e^{(p-iq)\left[\frac{1}{2}\log 2 - i\pi/4\right]}}$$

$$= \frac{e^{\frac{p}{2}\log 2 + \frac{pi\pi}{4} + \frac{iq}{2}\log 2 - \frac{q\pi}{4}}}{e^{\frac{p}{2}\log 2 - \frac{pi\pi}{4} - \frac{iq}{2}\log 2 - \frac{q\pi}{4}}}$$

$$= e^{\cancel{\frac{p}{2}\log 2} + \frac{pi\pi}{4} + \frac{iq\log 2}{2} - \cancel{\frac{q\pi}{4}} - \cancel{\frac{p}{2}\log 2} + \frac{pi\pi}{4} + \frac{iq}{2}\log 2 + \cancel{\frac{q\pi}{4}}}$$

$$= e^{pi\pi/2 + iq\log 2} = e^{i\left(\frac{p\pi}{2} + q\log 2\right)}$$

let $x = \frac{p\pi}{2} + i\log 2$

$$= e^{ix}$$

$$\therefore \quad \alpha + i\beta = e^{ix} \qquad \left[e^{ix} = \cos x + i\sin x\right]$$

$$= \cos x + i \sin x$$

Compare real and imaginary parts.

$$\cos x = \alpha \quad \text{—(i)}$$
$$\sin x = \beta \quad \text{—(ii)}$$

divide (ii) by (i)

$$\frac{\sin x}{\cos x} = \frac{\beta}{\alpha}$$

$$\tan x = \frac{\beta}{\alpha}$$

$$x = \tan^{-1}\frac{\beta}{\alpha}$$

$$\boxed{\frac{p\pi}{2} + q\log 2 = \tan^{-1}\frac{\beta}{\alpha}}$$