Plane Geometry

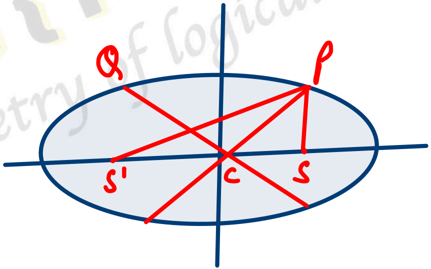
Ellipse

Prove that the product of focal distances of an extremity of a semi-diameter of an ellipse is equal to the square of conjugate semi diameter.

Proof

Let given ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



P and Q are extremities of conjugate semi diameter.

So $P(a\cos\theta, b\sin\theta)$

$Q(-a\sin\theta, b\cos\theta)$

$$PS' = a + ex_1$$

$$= a + e(a\cos\theta)$$

$$= a(1 + e\cos\theta)$$

$$PS = a - ex_1$$

$$= a - e(a\cos\theta)$$

$$= a[1 - e\cos\theta]$$

$$PS' \cdot PS = a(1+e\cos\theta) \cdot a(1-e\cos\theta)$$

$$= a^2(1-e^2\cos^2\theta)$$

$$= a^2 - a^2e^2\cos^2\theta$$

$$= a^2 - (a^2-b^2)\cos^2\theta$$

$$= a^2 - a^2\cos^2\theta + b^2\cos^2\theta$$

$$= a^2(1-\cos^2\theta) + b^2\cos^2\theta$$

$$PS' \cdot PS = a^2\sin^2\theta + b^2\cos^2\theta \quad \text{—} \; (1)$$

$$(CQ)^2 = \left(\sqrt{(-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2}\right)^2$$

$Q = (-a\sin\theta, b\cos\theta)$

$C = (0, 0)$

$$= a^2\sin^2\theta + b^2\cos^2\theta \quad \text{--- (ii)}$$

from (i) & (ii)

$PS' \cdot PS = (CQ)^2$

Hence Proved.