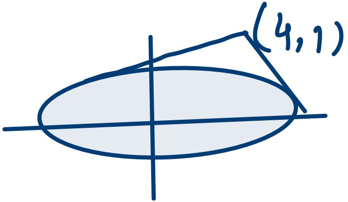
Plane Geometry

Ellipse

Find the equation of tangents drawn from the point $(4,1)$ to the ellipse $x^2 + 2y^2 = 6$ and prove that the angle between them is $\tan^{-1}\frac{3}{2}$



Sol: Given ellipse is

$$x^2 + 2y^2 = 6$$

$$\frac{x^2}{6} + \frac{2y^2}{6} = 1$$

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \text{---} \ (1)$$

eq. of tangent at (1)

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$y = mx + \sqrt{6m^2+3} \quad \text{— (ii)}$$

eq. (ii) Passes through (4, 1) (given)

$$1 = 4m + \sqrt{6m^2+3}$$

$$1 - 4m = \sqrt{6m^2+3}$$

sq. both side

$$(1-4m)^2 = 6m^2+3$$

$$1 + 16m^2 - 8m - 6m^2 - 3 = 0$$

$$10m^2 - 8m - 2 = 0$$

$$5m^2 - 4m - 1 = 0$$

$$5m^2 - 5m + m - 1 = 0$$

$$5m(m-1) + 1(m-1) = 0$$

$$(5m+1)(m-1) = 0$$

$$5m + 1 = 0 \quad 4 \quad m - 1 = 0$$

$$m = -1/5,\ 1.$$

$$m_1 = \frac{-1}{5} \qquad m_2 = 1.$$

for $m = -1/5$

eq. of tangent is

$$y = mx + \sqrt{6m^2 + 3}$$

$$y = \frac{-1}{5}x + \sqrt{6 \times \frac{1}{25} + 3}$$

$$y + \frac{1}{5}x = \sqrt{\frac{6 + 75}{25}}$$

$$y + \frac{1}{5}x = \sqrt{\frac{81}{25}}$$

$$y + \frac{1}{5}x = \frac{9}{5}$$

$$y + \frac{1}{5}x - \frac{9}{5} = 0$$

$$x + 5y - 9 = 0$$

for $m = 1$

$$y = x + \sqrt{6+3}$$

$$y - x = \sqrt{9}$$

$$y - x = 3$$

$$y - x - 3 = 0$$

$$x - y + 3 = 0$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{\frac{-1}{5} - 1}{1 + \left(\frac{-1}{5}\right)(1)}\right|$$

$$= \left|\frac{\frac{-1-5}{5}}{1 - \frac{1}{5}}\right| = \left|\frac{\frac{-6}{\cancel{5}}}{\frac{4}{\cancel{5}}}\right|$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\frac{3}{2}$$

Hence Proved.