Plane Geometry

Ellipse

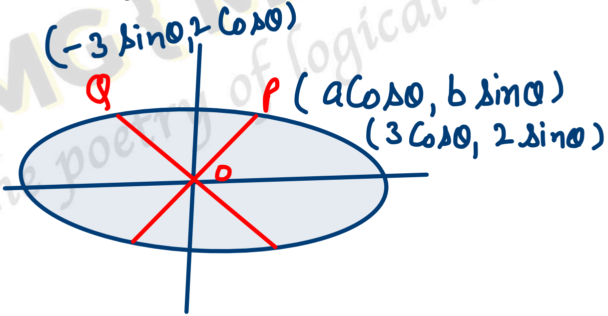
Find the minimum angle between a pair of conjugate diameters of the ellipse

$$4x^2 + 9y^2 = 36$$

Given ellipse is

$$4x^2 + 9y^2 = 36$$

$$\frac{4x^2}{36} + \frac{9y^2}{36} = 1$$



$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$a^2 = 9 \qquad b^2 = 4$$

$$a = 3 \qquad b = 2$$

$$P(a\cos\theta, b\sin\theta) \quad Q(-a\sin\theta, b\cos\theta)$$

$$P(3\cos\theta, 2\sin\theta) \quad Q(-3\sin\theta, 2\cos\theta)$$

$$\therefore \text{slope of } OP = \frac{2\sin\theta - 0}{3\cos\theta - 0} = \frac{2}{3}\frac{\sin\theta}{\cos\theta}$$

$$m_1 = \frac{2}{3}\tan\theta$$

slope of OQ $\quad m_2 = \frac{2\cos\theta - 0}{-3\sin\theta - 0} = -\frac{2}{3}\cot\theta$

Let ϕ be the angle b/w OP & OQ.

$$\tan\phi = \frac{\frac{2}{3}\tan\theta + \frac{2}{3}\cot\theta}{1 + \frac{2}{3}\tan\theta\left(-\frac{2}{3}\cot\theta\right)}$$

$$= \frac{\frac{2}{3}(\tan\theta + \cot\theta)}{1 - \frac{4}{9}}$$

$$= \frac{2}{\cancel{3}} \times \frac{\cancel{9}^{3}}{5}\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{6}{5}\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$\tan\phi = \frac{6}{5}\left(\frac{2}{2\sin\theta\cos\theta}\right) = \frac{6}{5} \times \frac{2}{\sin 2\theta}$$

$$\frac{12}{5}\,\frac{1}{\sin 2\theta}$$

$$\tan\phi = \frac{12}{5}$$

$$\phi = \tan^{-1}\left(\frac{12}{5}\right)$$