Trigonometry and Matrices

Function of a complex variable

Prove that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \text{-----} = \frac{\pi}{2\sqrt{2}}$

Sol. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \text{----}$

Let $x = \sqrt{i}$

$$\tan^{-1}\sqrt{i} = \sqrt{i} - \frac{(\sqrt{i})^3}{3} + \frac{(\sqrt{i})^5}{5} - \frac{(\sqrt{i})^7}{7} + \frac{(\sqrt{i})^9}{9} - \text{---}$$

$$= \sqrt{i}\left[1 - \frac{i}{3} - \frac{1}{5} + \frac{i}{7} + \frac{1}{9} - \cdots\right]$$

$$\tan^{-1}\sqrt{i} = \sqrt{\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]}\left[1 - \frac{i}{3} - \frac{1}{5} + \frac{i}{7} + \frac{1}{9} - \cdots\right]$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{2}}\left[1 - \frac{i}{3} - \frac{1}{5} + \frac{1}{7}i + \frac{1}{9} - \cdots\right]$$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(1 - \frac{i}{3} - \frac{1}{5} + \frac{1}{7}i + \frac{1}{9} - \cdots\right]$$

$$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(1 - \frac{i}{3} - \frac{1}{5} + \frac{1}{7}i + \frac{1}{9} - \cdots\right]$$

$$= \frac{1}{\sqrt{2}}(1+i)\left(1 - \frac{i}{3} - \frac{1}{5} + \frac{i}{7} + \frac{1}{9} + \cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 - \frac{i}{3} - \frac{1}{5} + \frac{i}{7} + \frac{1}{9} + \cdots + i + \frac{1}{3} - \frac{i}{5} - \frac{1}{7} + \frac{i}{9} - \cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots + i\left[1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots\right]\right]$$

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$$\tan^{-1}\sqrt{i} = \tan^{-1}\sqrt{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}$$

$$= \tan^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}$$

$$= \tan^{-1}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= x + iy \quad \text{--- (II)}$$

also $\tan^{-1}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = x - iy$ — (III)

Add (ii) & (iii)

$$x + i\cancel{y} + x - i\cancel{y} = \tan^{-1}\left(\cos\frac{\pi}{4} + i\sin\pi/4\right) + \tan^{-1}\left(\cos\pi/4 - i\sin\pi/4\right)$$

$$\frac{\tan^{-1}A + \tan^{-1}B}{} = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$2x = \tan^{-1}\left\{\frac{\cos\frac{\pi}{4} + \cancel{i\sin\frac{\pi}{4}} + \cos\pi/4 - \cancel{i\sin\frac{\pi}{4}}}{1-\left(\cos\frac{\pi}{4} + i\sin\pi/4\right)\left(\cos\pi/4 - i\sin\pi/4\right)}\right\}$$

$$2x = \tan^{-1}\left[\frac{2\cos\pi/4}{1-\left(\cos^2\pi/4 - i^2\sin^2\pi/4\right)}\right]$$

$$2x = \tan^{-1}\left[\frac{2 \cdot 1/\sqrt{2}}{1-\left(\cos^2 \pi/4 + \sin^2 \pi/4\right)}\right]$$

$$= \tan^{-1}\left[\frac{2/\sqrt{2}}{1-1}\right] = \tan^{-1}\infty = \pi/2$$

$$2x = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

x is real part of $\tan^{-1}\sqrt{i}$

$\therefore$ from ①

$$x = \frac{1}{\sqrt{2}}\left[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - - \dots\right]$$

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}}\left[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - - \dots\right]$$

$$\frac{\sqrt{2}\,\pi}{4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - - \dots$$

$$\frac{\sqrt{2}\,\pi}{2\times 2} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - \cdots$$

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - \cdots = \frac{\pi}{2\sqrt{2}}$$