Plane Geometry

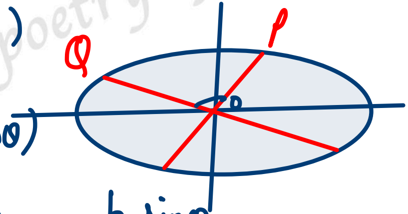
Ellipse

Show that the minimum angle between a pair of conjugate diameters of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \textbf{is} \quad \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$$

$P\ (a\cos\theta,\ b\sin\theta)$

$Q\ (-a\sin\theta,\ b\cos\theta)$



slope of OP $(m_1) = \dfrac{b\sin\theta}{a\cos\theta}$ —(1)

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Slope of OQ $m_2 = \dfrac{b\cos\theta}{-a\sin\theta} = \dfrac{-b}{a}\dfrac{\cos\theta}{\sin\theta}$ — (ii)

Let ϕ be angle b/w diameters.

$$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\phi = \frac{\left|\dfrac{b\sin\theta}{a\cos\theta} + \dfrac{b\cos\theta}{a\sin\theta}\right|}{1 + \left(\dfrac{b\sin\theta}{a\cos\theta}\right)\left(\dfrac{-b\cos\theta}{a\sin\theta}\right)}$$

$$\tan\phi = \frac{\frac{b}{a}\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)}{1 - \frac{b^2}{a^2}}$$

$$= \frac{\frac{b}{a}\,\frac{1}{\sin\theta\cos\theta}}{\frac{a^2-b^2}{a^2}}$$

$$= \frac{b}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{a^2-b^2}\left[\frac{1\cdot 2}{2\sin\theta\cos\theta}\right]$$

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$$\tan\phi = \frac{2ab}{a^2-b^2}\left[\frac{1}{\sin 2\theta}\right]$$

$$\tan\phi = \frac{2ab}{a^2-b^2} \qquad \left[\because \sin 2\theta = 1\right]$$

$$\phi = \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$$