Plane Geometry

Pair of straight lines

Prove that the product of the perpendiculars let fall from the point (x', y') upon the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a-b)^2 + 4h^2}}$$

Sol. Given eq. is

$$ax^2 + 2hxy + by^2 = 0 \quad \text{— (1)}$$

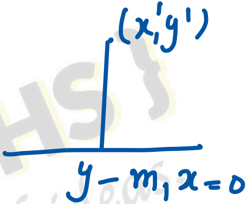
Let eq. (1) represent two lines

$$y - m_1 x = 0 \text{ and } y - m_2 x = 0$$

$$m_1 + m_2 = -\frac{2h}{b}$$

$$m_1 m_2 = \frac{a}{b}$$

Let p_1 is distance from (x', y') to $y - m_1 x = 0$



And p_2 is distance from (x', y') to $y - m_2 x = 0$

$$p_1 = \frac{y' - m_1 x'}{\sqrt{1 + m_1^2}} \text{ and } p_2 = \frac{y' - m_2 x'}{\sqrt{1 + m_2^2}}$$

$$p_1 p_2 = \frac{(y' - m_1 x')}{\sqrt{1 + m_1^2}} \cdot \frac{y' - m_2 x'}{\sqrt{1 + m_2^2}}$$

$$= \frac{(y' - m_1 x')(y' - m_2 x')}{\sqrt{(1 + m_1^2)(1 + m_2^2)}}$$

$$= \frac{y'^2 - m_2 x' y' - m_1 x' y' + m_1 m_2 x'^2}{\sqrt{1 + m_2^2 + m_1^2 + m_1^2 m_2^2}}$$

$$= \frac{y'^2 - (m_1 + m_2)x'y' + m_1 m_2 x'^2}{\sqrt{1 + (m_1+m_2)^2 - 2m_1 m_2 + m_1^2 m_2^2}}$$

$(m_1+m_2)^2$

$= m_1^2 + m_2^2 + 2m_1 m_2$

$$= \frac{y'^2 - \left(\frac{-2h}{b}\right)x'y' + \left(\frac{a}{b}\right)x'^2}{\sqrt{1 + \left(\frac{-2h}{b}\right)^2 - 2\left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2}}$$

$$= \frac{y'^2 + \frac{2h}{b}x'y' + \frac{a}{b}x'^2}{\sqrt{1 + \frac{4h^2}{b^2} - \frac{2a}{b} + \frac{a^2}{b^2}}}$$

$$= \frac{\cancel{\frac{1}{b}}\left(b y'^2 + 2h x'y' + a x'^2\right)}{\cancel{\frac{1}{b}}\sqrt{b^2 + 4h^2 - 2ab + a^2}}$$

$$= \frac{by'^2 + 2hx'y' + ax'^2}{\sqrt{(a-b)^2 + 4h^2}}$$

Hence Proved.