Calculus II

Asymptotes

Show that asymptote of the cubic curve

$x^3 - xy^2 - 2xy + 2x - y - 1 = 0$ cut the curve in atmost three points which lies on the line $3x - y - 1 = 0$

Sol.

Given Curve is

$$x^3 - xy^2 - 2xy + 2x - y - 1 = 0$$

$x = 1 \quad y = m.$

$$\phi_3(m) = 1 - m^2$$

$$\phi_2(m) = -2m.$$

$$\phi_1(m) = 2 - m.$$

Put $\phi_3(m) = 0$

$$1-m^2=0$$

$$(1-m)(1+m)=0$$

$$m=-1,1.$$

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}$$

$$=-\frac{(-2m)}{-2m}=-1.$$

$$c=-1.$$

Asymptotes are

$$y = mx + c$$

for $m = -1$ $\quad y = -x - 1$

$$y + x + 1 = 0$$

for $m = 1$. $\quad y = x - 1$

$$x - y - 1 = 0$$

Also Asymptote parallel to y-axis is $x = 0$

Asymptotes are

$$x = 0 \quad \text{--- (1)}$$

$$x + y + 1 = 0 \quad \text{--- (2)}$$

$$x - y - 1 = 0 \quad \text{--- (3)}$$

Multiply (i), (ii) & (iii)

$$x(x + y + 1)(x - y - 1) =$$

$$(x^2 + xy + x)(x - y - 1)$$

$$= x^3 - \cancel{x^2y} - \cancel{x^2} + \cancel{x^2y} - xy^2 - xy + \cancel{x^2} - xy - x$$

$$= x^3 - xy^2 - 2xy - x \quad \text{— (IV)}$$

Subtract (IV) from given curve.

$$\cancel{x^3} - \cancel{xy^2} - \cancel{2xy} + 2x - y - 1 - \cancel{x^3} + \cancel{xy^2} + \cancel{2xy} + x$$

$$3x - y - 1 = 0$$

$\Rightarrow$ Asymptotes cut the curve in points which lies on $3x - y - 1 = 0$

No. of points are $n(n-2)$

$3(3-2) = 3.$

$\Rightarrow$ Asymptotes cut the curve in three points which lies on

$3x - y - 1 = 0$ Hence Proved