Calculus II

Concavity and Convexity

Find the intervals in which the Curve $y = (\cos x + \sin x)e^x$ is Concave upward or downward in $(0, 2\pi)$

Find also the point of inflexion.

Sol

$$y = (\cos x + \sin x)e^x \quad \text{--- } (1)$$

$$\frac{dy}{dx} = (\cos x + \sin x)e^x + e^x(-\sin x + \cos x)$$

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$$= e^x\left[\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x\right]$$

$$\frac{dy}{dx} = 2e^x \cos x. \quad \text{--- (2)}$$

$$\frac{d^2y}{dx^2} = 2\left[e^x(-\sin x) + \cos x\, e^x\right]$$

$$= 2e^x(\cos x - \sin x) \quad \text{--- (3)}$$

$$\frac{d^3y}{dx^3} = 2\left[e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x\right]$$

$$= 2e^x(-\sin x - \cancel{\cos x} + \cancel{\cos x} - \sin x)$$

$$= 2e^x(-2\sin x)$$

$$\frac{d^3y}{dx^3} = -4e^x \sin x. \quad \text{—— } (4)$$

For Concave upward

$$\frac{d^2y}{dx^2} > 0$$

$$2e^x(\cos x - \sin x) > 0 \qquad [\text{from } ③]$$

$$\cos x - \sin x > 0 \qquad [\because 2e^x > 0]$$

$$\frac{1}{\sqrt{2}} \times \sqrt{2}\left[\cos x - \sin x\right] > 0$$

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$$\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right] > 0$$

$$\sqrt{2}\left[\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right] > 0$$

$$\sqrt{2}\sin\left(\frac{\pi}{4} - x\right) > 0$$

$$\sin\left(\frac{\pi}{4} - x\right) > 0$$

$\sin a \cos b - \cos a \sin b = \sin(a-b)$

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$$\sin\left(x - \pi/4\right) < 0$$

$$\sin\left(x - \pi/4\right) < 0$$

$$\Rightarrow \quad x - \frac{\pi}{4} \in (-\pi, 0) \cup (\pi, 2\pi)$$

and $x \in (0, 2\pi)$

$$\Rightarrow \quad x \in \left(\frac{-3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right) \quad \text{and } x \in (0, 2\pi)$$

$$\Rightarrow \quad x \in \left(0, \pi/4\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$$

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$\therefore$ The Given Curve is Concave upward in

$$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$$

for Concave downward.

$$\frac{d^2y}{dx^2} < 0$$

$$2e^x(\cos x - \sin x) < 0 \quad (\text{from } ③)$$

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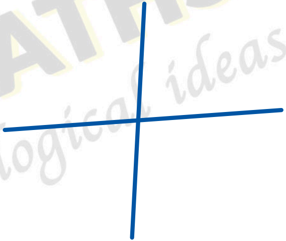
$$\cos x - \sin x < 0$$

$$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) > 0$$



$$\Rightarrow \quad x - \frac{\pi}{4} \in (0, \pi)$$

$$\Rightarrow \quad x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

$\therefore$ The Given Curve is Concave downward

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in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

for Point of inflexion.

$$\frac{d^2y}{dx^2} = 0$$

$$2e^x(\cos x - \sin x) = 0 \qquad [\text{from } 3]$$

$$\cos x - \sin x = 0$$

$$\sin x = \cos x$$

$$\tan x = 1.$$

$$\tan x = \tan\frac{\pi}{4},\ \tan\frac{5\pi}{4}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}.$$

from ④ $\quad \dfrac{d^3y}{dx^3} = -4e^x \sin x$

for $x = \pi/4$ $\quad \dfrac{d^3y}{dx^3} = -4e^{\pi/4}\sin\pi/4 = \dfrac{-4}{\sqrt{2}}e^{\pi/4} \neq 0$

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for $x = \frac{5\pi}{4}$ $\quad \frac{d^3y}{dx^3} = -4e^{5\pi/4} \sin\frac{5\pi}{4}$

$$= -4e^{5\pi/4}\left(-\frac{1}{\sqrt{2}}\right) \neq 0$$

$\frac{d^3y}{dx^3} \neq 0$ for $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\Rightarrow$ $\frac{\pi}{4}, \frac{5\pi}{4}$ are point of inflexion.

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for $x = \frac{\pi}{4}$

$$y = (\cos x + \sin x)\, e^{x}$$

$$= \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) e^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) e^{\pi/4} = \frac{2}{\sqrt{2}}\, e^{\pi/4}$$

$$= \sqrt{2}\, e^{\pi/4}.$$

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When $x = 5\pi/4$.

$$y = \left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right)e^{5\pi/4}$$

$$= \left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)e^{5\pi/4} = \frac{-2}{\sqrt{2}}e^{5\pi/4}$$

$$= -\sqrt{2}\,e^{5\pi/4}$$

$\therefore$ Given curve has point of inflexion at $\left(\pi/4, \sqrt{2}\,e^{\pi/4}\right)$, $\left(5\pi/4, -\sqrt{2}\,e^{5\pi/4}\right)$ Ans.

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