THEORY OF EQUATIONS

Prove that in an equation with rational coeff, irrational roots occur in conjugate pairs.

Proof

Let $f(x) = 0$ is given polynomial.

Now let $\alpha + \sqrt{\beta}$ is an irrational roots of $f(x)$

$\therefore\ f(\alpha + \sqrt{\beta}) = 0 \quad -(1)$

also $[x-(\alpha+\sqrt{\beta})]$ is factor of $f(x)$ — (11)

We have to prove that $\alpha-\sqrt{\beta}$ is also root of $f(x)$ s.t

$[x-(\alpha-\sqrt{\beta})]$ is factor of $f(x)$

Now $[x-(\alpha+\sqrt{\beta})][x-(\alpha-\sqrt{\beta})]$

$$= \{(x-\alpha)-\sqrt{\beta}\}[(x-\alpha)+\sqrt{\beta}]$$

$$= (x-\alpha)^2-(\sqrt{\beta})^2 = (x-\alpha)^2-\beta.$$

Divide $f(x)$ by $(x-\alpha)^2 - \beta$

Let Quotient is $Q(x)$ and Remeinder $Rx + S$.

$$f(x) = [(x-\alpha)^2 - \beta]\,Q(x) + Rx + S \quad \text{(iii)}$$

Now Put $x = \alpha + \sqrt{\beta}$.

$$f(\alpha + \sqrt{\beta}) = [(\cancel{\alpha} + \sqrt{\beta} - \cancel{\alpha})^2 - \beta]\,Q(\alpha + \sqrt{\beta}) + R(\alpha + \sqrt{\beta}) + S$$

$0 = R\alpha + R\sqrt{\beta} + S.$ [from (1)]

$\Rightarrow R\alpha + S = 0$

and $R\sqrt{\beta} = 0$

$R = 0$

Put $R = 0$ in $R\alpha + S = 0$

$\underline{\underline{S = 0}}$

$\therefore$ The remainder $Rx + S = 0.$

from (iii)

$$f(x) = \left[(x-\alpha)^2 - \beta\right] Q(x)$$

$$f(x) = \left[x - (\alpha + \sqrt{\beta})\right]\left[x - (\alpha - \sqrt{\beta})\right] Q(x)$$

$\therefore$ $(x - (\alpha - \sqrt{\beta}))$ is a factor of $f(x)$

$\Rightarrow$ $\alpha - \sqrt{\beta}$ is also root of $f(x)$

Hence whenever $\alpha + \sqrt{\beta}$ is root of $f(x)$

$\alpha - \sqrt{\beta}$ will be root of $f(x)$