THEORY OF EQUATIONS

Prove that in an equation with real Coeff., imaginary roots (i.e. non-real complex) always occur in conjugate pairs.

Proof :- Let $f(x) = 0$ is given eq.

Let $\alpha + i\beta$ is root of $f(x)$ where $\beta \neq 0$

We have to prove that $\alpha - i\beta$ is also root of $f(x)$

$[x - (\alpha - i\beta)]$ is factor of $f(x)$

Now $\alpha + i\beta$ is root of $f(x)$

$\therefore \quad [x - (\alpha + i\beta)]$ is factor of $f(x)$

also $\quad f(\alpha + i\beta) = 0 \quad$ —— ①

$$[x - (\alpha + i\beta)][x - (\alpha - i\beta)] = [(x-\alpha) - i\beta][(x-\alpha) + i\beta]$$

$$= (x-\alpha)^2 - i^2\beta^2$$

$$= (x-\alpha)^2 + \beta^2$$

Now divide $f(x)$ by $(x-\alpha)^2+\beta^2$

Let Quotient $Q(x)$ and

Remainder is $Rx+S$

$$f(x) = \left[(x-\alpha)^2+\beta^2\right] Q(x) + Rx + S \quad \text{--- (II)}$$

from (I)

$$f(\alpha+i\beta) = 0$$

Put $x = \alpha+i\beta$ in (II)

$$f(\alpha+i\beta) = [(\cancel{\alpha}+i\beta-\cancel{\alpha})^2+\beta^2]\,Q(x) + R(\alpha+i\beta) + S$$

$$0 = R\alpha + Ri\beta + S$$

$$0 + i0 = R\alpha + S + Ri\beta$$

equate real and imaginary Parts

$$R\alpha + S = 0 \quad \text{--- (iii)}$$

$$R\beta = 0$$

$$R = 0$$

Put $R=0$ in (11)

$S=0.$

Remainder $= Rx+S=0$

Put $Rx+S=0$ in (11)

$$f(x) = \left[(x-\alpha)^2+\beta^2\right]Q(x)$$

$$= \left[x-(\alpha+i\beta)\right]\left[x-(\alpha-i\beta)\right]Q(x)$$

$\Rightarrow \left(x-(\alpha-i\beta)\right)$ is factor of $f(x)$

$\therefore$ $\alpha - i\beta$ is root of $f(x)$

$\therefore$ whenever $\alpha + i\beta$ is root of $f(x)$

$\alpha - i\beta$ is also root of $f(x)$

Hence imaginary roots of eq. with real coeff always occur in Conjugate Pairs.

Hence Proved