

# Calculus

## Successive Differentiation : Important Questions

$$y = \sin(m \sin^{-1} x) \text{ Prove that}$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Deduce that  $y_n(0) = \begin{cases} 0 & n \text{ is even} \\ m(1^2-m^2)(3^2-m^2)(5^2-m^2)\dots & \text{when } n \text{ is odd.} \end{cases}$

Sol.

$$y = \sin(m \sin^{-1} x) \quad \text{--- (i)}$$

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{--- (ii)}$$

$$\sqrt{1-x^2} y_1 = m \cos(m \sin^{-1} x)$$

∴ both side

$$(1-x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x) \quad \cos^2 \theta$$

$$= m^2 (1 - \sin^2(m \sin^{-1} x)) \quad = 1 - \sin^2 \theta$$

$$(1-x^2) y_1^2 = m^2 (1-y^2)$$

diff. both side again

$$(1-x^2) 2y_1 y_2 + y_1^2 (-2x) = m^2 (-2y y_1)$$

$$\cancel{y_1} [(1-x^2) y_2 - x y_1] = m^2 (-2 \cancel{y} \cancel{y_1})$$

$$(1-x^2) y_2 - x y_1 + m^2 y = 0 \quad - \textcircled{iii}$$

$$\Rightarrow y_{n+2} (1-x^2) + n C_1 y_{n+1} (-2x)$$

$$+ n C_2 y_n (-2) - [y_{n+1} x + n C_1 y_n (1)] + m^2 y_n = 0$$

### 1.2 LEIBNITZ'S THEOREM

If  $u$  and  $v$  are functions of  $x$  such that their  $n^{\text{th}}$  derivatives exist, then the  $n^{\text{th}}$  derivative of their product is given by

$$(u v)_n = u_n v + n C_1 u_{n-1} v_1 + n C_2 u_{n-2} v_2 + \dots + n C_r u_{n-r} v_r + \dots + u v_n$$

where  $u_r$  and  $v_r$  represent  $r^{\text{th}}$  derivatives of  $u$  and  $v$  respectively.

$$\Rightarrow (1-x^2)y_{n+2} + n(-2x)y_{n+1} + (n^2-n)(-1)y_n - y_{n+1}x - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n + \cancel{ny_n} - \cancel{ny_n} + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$$

- (17)

Put  $x=0$  in (i), (ii), (iii) & (iv).

$$y(0) = 0$$

$$y_1(0) = m.$$

$$y_2(0) = 0.$$

$$y_{n+2} = (n^2 - m^2) y_n. \quad \text{--- (5)}$$

Put  $n=1, 2, 3, 4, 5 \dots$  in (5).

$$y_3 = (1^2 - m^2) m$$

$$y_4 = (2^2 - m^2) y_2(0) = 0$$

$$y_5 = (3^2 - m^2) y_3(0) = m(1^2 - m^2)(3^2 - m^2)$$

$$y_6 = (4^2 - m^2) y_4(0) = 0$$

from above values we conclude that

$$y_n(0) = \begin{cases} 0 & n \text{ is even.} \\ m(1^2 - m^2)(3^2 - m^2) \\ (5^2 - m^2) \dots ((n-2)^2 - m^2) \end{cases} \quad \left. \vphantom{\begin{cases} 0 \\ m(1^2 - m^2)(3^2 - m^2) \\ (5^2 - m^2) \dots ((n-2)^2 - m^2) \end{cases}} \right\} n \text{ is odd.}$$

Hence Proved.



$$\textcircled{2} \quad x = \cos\left(\frac{1}{m} \log y\right)$$

Prove that

$$(1-x^2) y_{n+2} - (2n+1)x y_{n+1} - (n^2+m^2)y_n = 0$$

Sol  
=

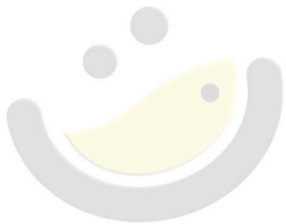
$$x = \cos\left(\frac{1}{m} \log y\right)$$

$$\cos^{-1} x = \frac{1}{m} \log y.$$

$$m \cos^{-1} x = \log y.$$

$$e^{m \cos^{-1} x} = e^{\log y}.$$

$$e^{m \cos^{-1} x} = y.$$



OMG { MATHS }

The poetry of logical ideas.