Calculus

Successive Differentiation : Important Questions

$y = \sin(m \sin^{-1} x)$ Prove that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - m^2) y_n = 0$$

Deduce that $y_n(0) = \begin{cases} 0 & n \text{ is even} \\ m(1^2 - m^2)(3^2 - m^2)(5^2 - m^2) \cdots & \text{when } n \text{ is odd.} \end{cases}$

Sol.

$$y = \sin(m \sin^{-1} x) \quad \text{--- (i)}$$

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{--- (ii)}$$

$$\sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$$

sq. both side

$$(1-x^2)\, y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$\cos^2\theta$

$= 1-\sin^2\theta$

$$= m^2\left(1-\sin^2(m\sin^{-1}x)\right)$$

$$(1-x^2)\, y_1^2 = m^2(1-y^2)$$

diff. both side again

$$(1-x^2)\,2y_1y_2 + y_1^2(-2x) = m^2(-2yy_1)$$

$$\cancel{2}\cancel{y_1}\left[(1-x^2)y_2 - xy_1\right] = m^2(-\cancel{2}y\cancel{y_1})$$

$$(1-x^2)y_2 - xy_1 + m^2y = 0 \quad \text{— (ii)}$$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

$$\Rightarrow y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - \left[y_{n+1}x + {}^nC_1\, y_n(1)\right] + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + n(-2x)y_{n+1} + (n^2-n)(-1)y_n$$

$$- y_{n+1}x - ny_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n + n\cancel{y_n} - n\cancel{y_n}$$

$$+ m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

— IV

Put $x=0$ in (i), (ii), (iii) & (iv).

$$y(0) = 0$$

$$y_1(0) = m.$$

$$y_2(0) = 0.$$

$$y_{n+2} = (n^2 - m^2)\, y_n. \quad \text{—— (5)}$$

Put $n = 1, 2, 3, 4, 5 \ldots\ldots$ in (5).

$$y_3 = (1^2 - m^2)\, m$$

$$y_4 = (2^2 - m^2)\, y_2(0) = 0$$

$$y_5 = (3^2 - m^2)\, y_3(0) = m\,(1^2 - m^2)(3^2 - m^2)$$

$$y_6 = (4^2 - m^2)\, y_4(0) = 0$$

from above values we conclude that

$$y_n(0) = \begin{cases} 0 & n \text{ is even.} \\ m\,(1^2 - m^2)(3^2 - m^2)(5^2 - m^2) \ldots ((n-2)^2 - m^2) & n \text{ is odd.} \end{cases}$$

Hence Proved.

(2) $x = \cos\left(\frac{1}{m}\log y\right)$

Prove that

$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (n^2+m^2)\,y_n = 0$

Sol

$x = \cos\left(\frac{1}{m}\log y\right)$

$\cos^{-1}x = \frac{1}{m}\log y.$

$$m \cos^{-1}x = \log y.$$

$$e^{m \cos^{-1}x} = e^{\log y}.$$

$$e^{m \cos^{-1}x} = y.$$