EIGEN VALUES AND CAYLEY-HAMILTON THEOREM

Most Important Theorem

State and Prove Cayley-Hamilton Theorem

Statement : Every Square Matrix satisfies its Characteristics Equation

Proof

Let A is $n \times n$ square Matrix.

Now $|A - \lambda I| = 0$ is ch. eq. of A

$$|A - \lambda I| = a_0 + a_1 \lambda + a_2 \lambda^2 + \text{———} + a_n \lambda^n = 0 \quad \text{— (1)}$$

T.P. $a_0 + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$

$$A \cdot \operatorname{adj} A = |A| I$$

$$(A - \lambda I)\, \operatorname{Adj}(A - \lambda I) = |A - \lambda I| I$$

Let $\operatorname{Adj}(A - \lambda I) = b_0 + b_1 \lambda + b_2 \lambda^2 + \cdots + b_{n-1}\lambda^{n-1}$

$\left[\,|\operatorname{adj} A| = |A|^{n-1}\right]$

$$(A - \lambda I)\left(b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-2}\lambda^{n-2} + b_{n-1}\lambda^{n-1}\right)$$

$$= \left\{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n\right\} I \quad [\text{from } (1)]$$

Compare the like Coeff of powers of λ

$$A b_0 = a_0 I$$

$$A b_1 - b_0 = a_1 I$$

$$A b_2 - b_1 = a_2 I$$

$$\cdots\cdots$$

$$A b_{n-1} - b_{n-2} = a_{n-1} I$$

$$-b_{n-1} = a_n I$$

Now Multiply above eq. with $I, A, A^2 \ldots \ldots A^n$ respectively.

$$A b_0 = a_0 I$$

$$A^2 b_1 - A b_0 = A a_1 I$$

$$A^3 b_2 - A^2 b_1 = A^2 a_2 I$$

$$\vdots$$

$$A^n b_{n-1} - A^{n-1} b_{n-2} = A^{n-1} a_{n-1} I$$

$$- A^n b_{n-1} = A^n a_n I$$

Now add above eqs.

$$O = a_0 + A a_1 + A^2 a_2 + A^3 a_3 + \cdots \cdots A^{n-1} a_{n-1} + A^n a_n$$

which is the required result

Hence Proved.