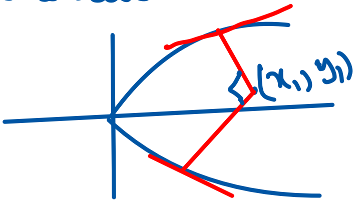
Plane Geometry

Parabola

Important Questions (PYQ)

Find the locus of the point such that two of the normals drawn through it to the parabola $y^2 = 4ax$ are perpendicular to each other.



Sol. :- $y^2 = 4ax$ (Given)

Let (x_1, y_1) is given Point

The equation of normal through (x_1, y_1)

$$y = mx - 2am - am^3$$

$$y_1 = mx_1 - 2am - am^3$$

$$am^3 + 2am - mx_1 + y_1 = 0$$

$$am^3 + (2a - x_1)m + y_1 = 0 \quad \text{--- (1)}$$

Let m_1, m_2, m_3 are roots of (1)

$$m_1 m_2 m_3 = -\frac{y_1}{a} \quad \text{--- (II)}$$

Also two normals are $\perp$ to each other (Given)

$$\therefore \quad m_1 m_2 = -1$$

from (II) $\quad (-1)\, m_3 = -\frac{y_1}{a}$

$$m_3 = \frac{y_1}{a} \quad \text{--- } (\star)$$

m_3 is root of eq. (I).

$$a(m_3)^2 + (2a - x_1)\, m_3 + y_1 = 0$$

$$a\left(\frac{y_1}{a}\right)^3 + (2a - x_1)\frac{y_1}{a} + y_1 = 0$$

$$\frac{y_1^3}{a^2} + (2a - x_1)\frac{y_1}{a} + y_1 = 0$$

$$\frac{y_1}{a}\left(\frac{y_1^2}{a} + (2a - x_1) + a\right) = 0$$

$$y_1^2 + (2a - x_1)a + a^2 = 0$$

$$y_1^2 + 2a^2 - ax_1 + a^2 = 0$$

$$y_1^2 - ax_1 + 3a^2 = 0$$

Locus of (x_1, y_1) is

$$\boxed{y^2 - ax + 3a^2 = 0}$$

which is required locus.

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② Prove that the locus of poles of chords which are normal to the parabola $y^2 = 4ax$ is the curve

$$y^2(x+2a) + 4a^3 = 0$$

Sol

$y^2 = 4ax$ is the Parabola.

eq. of normal to the Parabola.

$$y = mx - 2am - am^3 \quad -(1)$$

Let (x_1, y_1) be the pole

eq. of Polar at (x_1, y_1)

$$yy_1 = 2a(x + x_1)$$

$$yy_1 = 2ax + 2ax_1 \quad -(2)$$

ie. (1) and (2) represent the same lines.

$\therefore \quad \dfrac{y_1}{1} = \dfrac{2a}{m} = \dfrac{2ax_1}{-2am-am^3}$

$my_1 = 2a$

$m = \dfrac{2a}{y_1}$

$2a(-2am-am^3) = 2amx_1$

$-2am - am^3 = mx_1$

$mx_1 + 2am + am^3 = 0$

$x_1 + 2a + am^2 = 0$

$$x_1 + 2a + a\left(\frac{2a}{y_1}\right)^2 = 0$$

$$x_1 + 2a + a \times \frac{4a^2}{y_1^2} = 0$$

$$y_1^2 x_1 + 2a y_1^2 + 4a^3 = 0$$

$$4a^3 + (x_1 + 2a) y_1^2 = 0$$

$\therefore$ Locus of (x_1, y_1) is

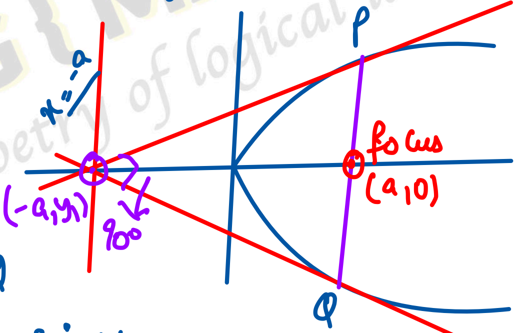
$(x + 2a) y^2 + 4a^3 = 0$ Hence Proved

③ Prove that in a Parabola chord of Contact of tangents at right angles passes through the focus.

Sol Let Parabola

$y^2 = 4ax$

Tangent at P & Q intersect at right angle (Given)



$\therefore$ Point of intersection R lies on directrix

Let R is $(-a, y_1)$

Chord of Contact of R $(-a, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

$$yy_1 = 2a(x - a) \quad \text{--- (1)}$$

(1) Passes through focus $(a, 0)$ if

$$(0)y_1 = 2a(a - a)$$

$0 = 0$ which is true.

Hence chord of contact passes through focus.

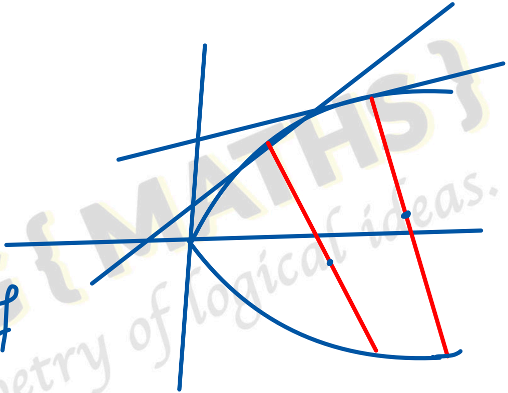
(4) Prove that the locus of middle points of normal chords of parabola $y^2 = 4ax$ is

$$\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a.$$

Sol Parabola is

$y^2 = 4ax$

eq. of Normal of Parabola.



$y = mx - 2am - am^3$ —①

Let (x_1, y_1) is mid-point of ①

eq. of Chord having mid-point (x_1, y_1)

$$yy_1 - 2a(x+x_1) = y_1^2 - 4ax_1$$

$$yy_1 - 2ax - 2ax_1 = y_1^2 - 4ax_1$$

$$yy_1 - 2ax = y_1^2 - 2ax_1$$

$$yy_1 = y_1^2 - 2ax_1 + 2ax \quad \text{--- (ii)}$$

(i) and (ii) represent the same line.

$$\therefore \frac{y_1}{1} = \frac{2a}{m} = \frac{y_1^2 - 2ax_1}{-2am - am^3}$$

$$y_1 m = 2a$$

$$m = \frac{2a}{y_1}$$

$$2a(-2am - am^3) = m(y_1^2 - 2ax_1)$$

$$\cancel{m}(-4a^2 - 2a^2m^2) = \cancel{m}(y_1^2 - 2ax_1)$$

$$-4a^2 - 2a^2 \cdot \left(\frac{2a}{y_1}\right)^2 = y_1^2 - 2ax_1$$

$$-4a^2 - \frac{8a^4}{y_1^2} = y_1^2 - 2ax_1$$

$$2ax_1 - 4a^2 = y_1^2 + \frac{8a^4}{y_1^2}$$

$$\cancel{2a}(x_1 - 2a) = \cancel{2a}\left(\frac{y_1^2}{2a} + \frac{4a^3}{y_1^2}\right)$$

$$x_1 - 2a = \frac{y_1^2}{2a} + \frac{4a^3}{y_1^2}$$

$\therefore$ locus of (x_1, y_1) is

$$x - 2a = \frac{y^2}{2a} + \frac{4a^3}{y^2}$$ Hence proved.