Plane Geometry

Pair Of Straight Lines

Full Chapter Revision and PYQ'S

* $(ax+by+c)(a'x+b'y+c')=0$ represents a pair of lines

$$ax+by+c=0$$
$$a'x+b'y+c'=0$$

* The second degree homogenous equation $ax^2+2hxy+by^2=0$

- a Pair of distinct lines through origin

$$h^2 - ab > 0$$

- a line or Coincide lines through origin

$$h^2 - ab = 0$$

- an empty set if $h^2 - ab < 0$

$ax^2 + 2hxy + by^2 = 0$ represent two lines through origin.

$y = m_1 x \qquad y = m_2 x.$

$$bm^2 + 2hm + a = 0$$

$$m_1 + m_2 = \frac{-2h}{b}$$

$x = 1$
$y = m.$

$$m_1 m_2 = \frac{a}{b}.$$

☆ θ between pair of lines

$$ax^2 + 2hxy + by^2 = 0$$

$$\boxed{\tan\theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|}$$

- The lines $ax^2 + 2hxy + by^2 = 0$ are perpendicular when $a + b = 0$

  Coeff. of x^2 + Coeff of y^2 = 0

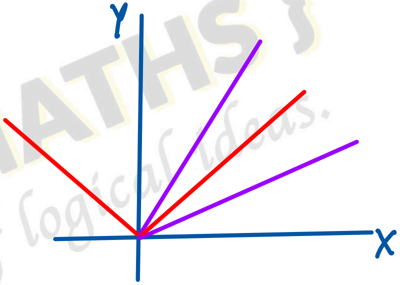
- The lines $ax^2 + 2hxy + by^2 = 0$ are Coincide when

  $$h^2 - ab = 0$$

✩ The Joint equation of straight lines Bisecting the angles between lines

$ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}.$$



General second degree equation.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent the straight lines

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

✡ $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent pair of lines then joint equation of lines through origin and parallel to these lines

$$ax^2 + 2hxy + by^2 = 0.$$

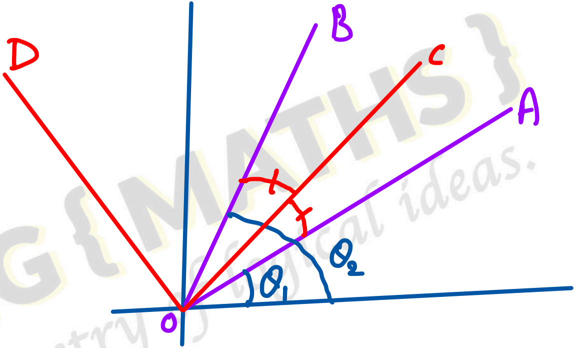
① Find the equation to the straight lines bisecting the angles between the straight lines given by $ax^2 + 2hxy + by^2 = 0$

Sol: Let OA and OB are two lines

$y = m_1 x$ and $y = m_2 x$

$$m_1 = \tan\theta_1$$

$$m_2 = \tan\theta_2$$



$$m_1 + m_2 = \frac{-2h}{b}$$

$$m_1 m_2 = \frac{a}{b}$$

$$\left.\begin{aligned} \tan\theta_1 + \tan\theta_2 &= \frac{-2h}{b} \\ \tan\theta_1 \tan\theta_2 &= a/b \end{aligned}\right] \quad -(1)$$

Let OC is internal and OD is External Bisector of $\angle AOB$.

Now

$$\angle AOC = \angle COB$$

$$\angle XOC - \theta_1 = \theta_2 - \angle XOC$$

$$2\angle XOC = \theta_1 + \theta_2$$

$$\angle XOC = \frac{\theta_1 + \theta_2}{2}$$

$\angle XOD = \angle XOC + \angle COD$

$$= \frac{\theta_1 + \theta_2}{2} + \pi/2.$$

[Angle between interior and exterior Bisector is always 90°]

Let θ be the angle made by bisectors with x-axis

either $\theta = \frac{\theta_1 + \theta_2}{2}$ or $\theta = \frac{\theta_1 + \theta_2}{2} + \pi/2$

either $2\theta = \theta_1 + \theta_2$ or $2\theta = \theta_1 + \theta_2 + \pi$

$$\tan 2\theta = \tan(\theta_1 + \theta_2)$$

$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{-\frac{2h}{b}}{1 - \frac{a}{b}}$$

[from ①]

$$= \frac{-2h}{b-a} = \frac{2h}{a-b}$$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b} \quad — (2)$$

equation of bisector is $y = \tan\theta \, x$.

$\tan\theta = y/x$

Put value of tanθ in ②.

$$\frac{2\,y/x}{1-y^2/x^2} = \frac{2h}{a-b}$$

$$\frac{y}{x\,\frac{(x^2-y^2)}{x^2}} = \frac{h}{a-b}$$

$$\frac{xy}{x^2-y^2} = \frac{h}{a-b}.$$

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

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(2) Show that the equation

$$x^2+2\sqrt{3}\,xy+3y^2-3x-3\sqrt{3}\,y-4=0$$

represent a pair of parallel lines. Find the distance between them.

Sol.

$$x^2+2\sqrt{3}\,xy+3y^2-3x-3\sqrt{3}\,y-4=0$$

$$x^2 + (2\sqrt{3}y - 3)x + 3y^2 - 3\sqrt{3}y - 4 = 0$$

$$x = \frac{-(2\sqrt{3}y - 3) \pm \sqrt{(2\sqrt{3}y - 3)^2 - 4(1)(3y^2 - 3\sqrt{3}y - 4)}}{2}$$

$$= \frac{-2\sqrt{3}y + 3 \pm \sqrt{12\cancel{y^2} + 9 - 12\cancel{\sqrt{3}y} - 12\cancel{y^2} + 12\cancel{\sqrt{3}y} + 16}}{2}$$

$$= \frac{-2\sqrt{3}y + 3 \pm \sqrt{25}}{2}$$

$$x = \frac{-2\sqrt{3}y + 3 \pm 5}{2}$$

$$x = \frac{-2\sqrt{3}y + 8}{2}, \quad \frac{-2\sqrt{3}y - 2}{2}$$

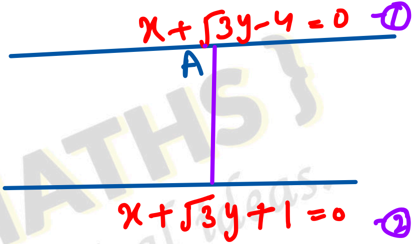
$$x + \sqrt{3}y - 4 = 0 \quad \text{--- (1)}, \qquad x + \sqrt{3}y + 1 = 0 \quad \text{--- (2)}$$

slope of 1 is $\frac{-1}{\sqrt{3}}$.

slope of 2 is $\frac{-1}{\sqrt{3}}$.

slope of 1 = slope of 2 = $\frac{-1}{\sqrt{3}}$

$\therefore$ lines 1 and 11 are parallel



Put $y=0$ in ①

$x = 4.$

Point A is $(4,0)$

$\therefore$ Distance from $(4,0)$ to ②

$$\frac{4+0+1}{\sqrt{1+3}} = \frac{5}{2} \text{ Ans.}$$

③ If p_1, p_2 are the lengths of perpendiculars drawn from the point $(-1, 2)$ to the pair of lines given by the equation

$$2x^2 - 5xy + 2y^2 + 3x - 3y + 1 = 0$$

Prove that $p_1 p_2 = \frac{12}{5}$

Sol.

$$2x^2 - 5xy + 2y^2 + 3x - 3y + 1 = 0$$

$$2x^2 + (-5y + 3)x + 2y^2 - 3y + 1 = 0$$

$$x = \frac{-(-5y+3) \pm \sqrt{(-5y+3)^2 - 4(2)(2y^2-3y+1)}}{4}$$

$$= \frac{5y-3 \pm \sqrt{25y^2+9-30y-16y^2+24y-8}}{4}$$

$$= \frac{5y-3 \pm \sqrt{9y^2-6y+1}}{4}$$

$$= \frac{5y-3 \pm \sqrt{(3y-1)^2}}{4}$$

$$= \frac{5y-3+3y-1}{4}, \frac{5y-3-3y+1}{4}$$

$$= \frac{8y-4}{4}, \frac{2y-2}{4}$$

$$= 2y-1, \frac{y-1}{2}$$

$x = 2y - 1$

$x - 2y + 1 = 0$ — ①

$2x - y + 1 = 0$ — ②

$\perp$ distance of ① from $(-1, 2)$

$$p_1 = \frac{|-1 - 4 + 1|}{\sqrt{1 + 4}} = \frac{4}{\sqrt{5}}$$

P_2 is $\perp$ distance of ② from $(-1,2)$

$$P_2 = \frac{|-2-2+1|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$$

$$P_1 P_2 = \frac{4}{\sqrt{5}} \times \frac{3}{\sqrt{5}} = \frac{12}{5} \text{ Ans.}$$

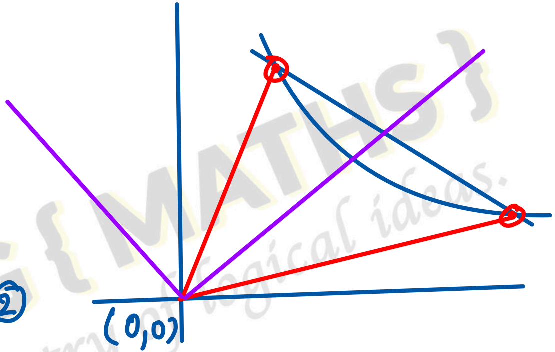
Hence Proved.

(4) Find the bisectors of the angles between the lines joining the origin to the point of intersection of the straight line $x - y = 2$ with the curve

$$5x^2 + 11xy - 8y^2 + 8x - 4y + 12 = 0$$

Sol. $x - y = 2 \quad -①$

$5x^2 + 11xy - 8y^2 + 8x - 4y + 12 = 0 \quad -②$



$x - y = 2$

$\frac{x}{2} - \frac{y}{2} = 1. \quad -③$

use ③ to make ② a homogeneous eq.

$$5x^2 + 11xy - 8y^2 + 8x\left(\frac{x}{2} - \frac{y}{2}\right) - 4y\left(\frac{x}{2} - \frac{y}{2}\right) + 12\left(\frac{x}{2} - \frac{y}{2}\right)^2 = 0$$

$$5x^2 + 11xy - 8y^2 + 4x^2 - 4xy - 2xy + 2y^2 + 12\left(\frac{x^2}{4} + \frac{y^2}{4} - \frac{xy}{2}\right) = 0$$

$$5x^2 + 11xy - 8y^2 + 4x^2 - 4xy - 2xy + 2y^2$$

$$+ 3x^2 + 3y^2 - 6xy = 0$$

$$12x^2 - 3y^2 - xy = 0$$

which is the joint equation of the lines through origin.

The equation of bisectors of the angles between the lines. is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

$$\frac{x^2-y^2}{12+3}=\frac{xy}{-1/2}$$

$$\frac{x^2-y^2}{15}=-2xy$$

$$x^2-y^2=-30xy$$

$12x^2-3y^2-xy=0$

$ax^2+2hxy+by^2=0$

$a=12 \quad b=-3$

$h=-1/2.$

$$x^2 - y^2 + 30xy = 0$$

which is the required equation.