Calculus

Limit and Continuity : Important Questions

Show that the function

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is Discontinous at $x = 0$

Sol.

$$\lim_{x \to 0^-} \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$x = 0 - h,\ h > 0$

$x \to 0^- \quad h \to 0$

$$\lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

$$\lim_{x \to 0^-} f(x) = -1 \quad \longrightarrow \quad (1)$$

$$\lim_{x \to 0^+} \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$x = 0 + h, \quad h > 0$

$x \to 0^+ \quad h \to 0$

$$\lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

Divide numerator and denominator with $e^{1/h}$.

$$\lim_{h \to 0} \frac{\frac{e^{1/h}}{e^{1/h}} - \frac{1}{e^{1/h}}}{e^{1/h}\left(e^{1/h} + 1\right)/e^{1/h}}$$

$$\lim_{h \to 0} \frac{1 - e^{-1/h}}{1 + e^{-1/h}} = \frac{1-0}{1+0} = 1.$$

$$\lim_{x \to 0^+} f(x) = 1. \quad \text{—} \; (ii)$$

From (i) & (ii)

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\therefore$ $f(x)$ is discontinous at $x=0$.

Hence Proved.

(2) Prove that $\lim_{x \to a} \frac{1}{x-a}$ does not exist.

(3) Show that $\lim_{x \to 0} \sin 1/x$ does not exist.