Hermitian and Skew Hermitian Matrices

Important Questions (PYQ)

Transpose of Matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Symmetric Matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} = A$$

$$A' = A$$

Symmetric Matrix

Skew - symmetric Matrix. $A' = -A$.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

$A' = -A$ Skew Symmetric Matrix

* Show that the Matrix $B'AB$ is symmetric or skew symmetric acording as A is symmetric or skew symmetric.

Proof: A is symmetric

$A' = A$ —(1)

$(B'AB)' = B'A'(B')'$

$$= \underline{B'}\ \underline{A'}\ \underline{B} \qquad \{\because (B')' = B\}$$

$$= B'AB \qquad [\text{from } ①]$$

$$(B'AB)' = \underline{\underline{B'AB}}$$

$\therefore$ $B'AB$ is symmetric.

<u>Case II</u> A is skew-symmetric.

$$A' = -A. \quad —②$$

$$(B'AB)' = B'A'(B')'$$

$$= B' A' B$$

$$= B'(-A)B \quad \{\text{from } 2\}$$

$$= -(B'AB)$$

$$(B'AB)' = -(B'AB)$$

Hence $B'AB$ is skew-symmetric.

Conjugate Matrix $\begin{bmatrix} 1 & -2i \\ 4+i & 2 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & 2i \\ 4-i & 2 \end{bmatrix}$$

Tranjugate Matrix

$$A = \begin{bmatrix} 1 & -2i \\ 4+i & 2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 4+i \\ -2i & 2 \end{bmatrix}$$

$$A^{\theta} = (\overline{A'}) = \begin{bmatrix} 1 & 4-i \\ 2i & 2 \end{bmatrix}$$

Is $(\bar{A})' = \overline{(A')}$ Justify your Answer.

Yes.

Justification

$$A = \begin{bmatrix} 1+i & 3i \\ 4 & 1-i \end{bmatrix}$$

L.H.S $(\bar{A})'$

$$\bar{A} = \begin{bmatrix} 1-i & -3i \\ 4 & 1+i \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 1-i & 4 \\ -3i & 1+i \end{bmatrix} \quad \text{--- } ①$$

$$A = \begin{bmatrix} 1+i & 3i \\ 4 & 1-i \end{bmatrix}$$

$$A' = \begin{bmatrix} 1+i & 4 \\ 3i & 1-i \end{bmatrix}$$

$$\overline{(A')} = \begin{bmatrix} 1-i & 4 \\ -3i & 1+i \end{bmatrix} \quad \text{— (ii)}$$

from (i) & (ii)

$$(\bar{A})' = \overline{(A')}$$

Hence Proved.

Hermitian Matrix

Square $\qquad A^{\theta} = A.$

$$A = \begin{bmatrix} 1 & 4+3i \\ 4-3i & 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 4-3i \\ 4+3i & 5 \end{bmatrix}$$

$$(\overline{A'}) = \begin{bmatrix} 1 & 4+3i \\ 4-3i & 5 \end{bmatrix} = A$$

Skew hermitian matrix

$$A^{\theta} = -A$$

$a_{12} = 4+3i$

$a_{21} = 4-3i$

$\bar{a}_{21} = 4+3i$

$a_{12} = \bar{a}_{21}$

$a_{ij} = \bar{a}_{ji}$

hermitiam $a_{ij} = \bar{a}_{ji}$

Prove that the necessary and sufficient condition for a matrix A to be hermitian is $A^{\theta} = A$

Proof

A is hermitian

$$a_{ij} = \bar{a}_{ji} \quad \text{--- (1)}$$

Now A is $n \times n$ matrix.

$\therefore A'$ is $n \times n$ matrix.

$\therefore (\overline{A'})$ is $n \times n$ matrix

$\therefore$ A^{θ} is $n \times n$ matrix.

$\therefore$ A^{θ} & A is of same type

<u>Now</u> $(i,j)^{th}$ element of A^{θ} = Complex Conjugate of $(i,j)^{th}$ element of A'

= Complex Conjugate of $(j,i)^{th}$ element of A

= Complex Conjugate of a_{ji}

$= \overline{\bar{a}_{ji}}$

$= a_{ij}$ from ①

$= (i,j)^{th}$ element of A.

$\Rightarrow$ $(i,j)^{th}$ element of $A^{\theta} = (i,j)^{th}$ element of A

$\Rightarrow$ $A^{\theta} = A$

Part II

$\overline{A^{\theta}} = A$

Let A is $m \times n$ matrix.

A' is $n \times m$ matrix

$A^{\theta} = (\overline{A'})$ is $n \times m$ matrix.

But $A^{\theta} = A$.

$\therefore$ A^{θ} & A is of same type.

$\therefore$ $\underline{\underline{n = m}}$.

$\Rightarrow$ A is $n \times n$ square matrix.

$$A^{\theta} = A$$

$\Rightarrow$ $(i,j)^{th}$ element of A^{θ} = $(i,j)^{th}$ element of A

$\Rightarrow$ Complex Conjugate of $(i,j)^{th}$ element of A'

= $(i,j)^{th}$ element of A

$\Rightarrow$ Complex Conjugate of $(j,i)^{th}$ element of A

= $(i,j)^{th}$ element of A

$$\bar{a}_{ji} = a_{ij}$$

$\Rightarrow$ A is Hermitian.

✲ A and B are hermitian. Show that $AB+BA$ is hermitian and $AB-BA$ is skew-hermitian.

Sol. A and B are hermitian (Given)

$$\therefore A^{\theta} = A \qquad B^{\theta} = B. \quad \text{---} (1)$$

$$(AB+BA)^{\theta} = (AB)^{\theta} + (BA)^{\theta}$$
$$= B^{\theta}A^{\theta} + A^{\theta}B^{\theta}$$

$$= BA + AB \quad [\text{from } ①]$$

$$= AB + BA$$

$$(AB + BA)^\theta = AB + BA$$

$\therefore$ $AB + BA$ is hermitian

$$(AB - BA)^\theta = (AB)^\theta - (BA)^\theta$$

$$= B^\theta A^\theta - A^\theta B^\theta$$

$$= BA - AB \quad [\text{from } ①]$$

$$= -(AB - BA)$$

$$(AB - BA)^{\theta} = -(AB - BA)$$

Hence $AB - BA$ is skew-hermitian matrix.

Q Show that every square matrix can be expressed in one and only one way as a sum of a hermitian and a skew-hermitian matrix.

Sol. Let A is a square matrix

$$A = \frac{A}{2} + \frac{A}{2}$$

$$= \frac{A}{2} + \frac{A}{2} + \frac{A^{\theta}}{2} - \frac{A^{\theta}}{2}$$

$$= \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$$A = P + Q$$

$$P = \frac{1}{2}\left(A + A^{\theta}\right)$$

$$P^{\theta} = \frac{1}{2}\left(A + A^{\theta}\right)^{\theta} = \frac{1}{2}\left(A^{\theta} + (A^{\theta})^{\theta}\right)$$

$$= \frac{1}{2}\left(A^{\theta} + A\right) = \frac{1}{2}\left(A + A^{\theta}\right)$$

$$= P.$$

$$\underbrace{P^{\theta} = P}$$

$\Rightarrow P$ is hermitian.

$$Q = \frac{1}{2}\left[A - A^{\theta}\right]$$

$$Q^{\theta} = \frac{1}{2}\left[A - A^{\theta}\right]^{\theta}$$

$$= \frac{1}{2}\left[A^{\theta} - (A^{\theta})^{\theta}\right]$$

$$= \frac{1}{2}\left(A^{\theta} - A\right) = -\frac{1}{2}\left(A - A^{\theta}\right) = -Q.$$

$$\underline{\underline{Q^{\theta} = -Q.}}$$ Skew - hermitian matrix

$$A = P + Q$$

P → hermitian, Q → skew hermitian

Hence a square matrix can be expressed in hermitian and skew hermitian matrix.

Let $A = R + S$

where R is hermitian and S is skew - hermitian

$$R^{\theta} = R \qquad S^{\theta} = -S.$$

$$A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta}$$

$$A^{\theta} = R - S. \quad \text{—(i)}$$

$$A = R + S \quad \text{—(ii)}$$

Add (i) & (ii)

$$A + A^{\theta} = 2R$$

$$\frac{A + A^{\theta}}{2} = R = P$$

Subtract (i) from (ii)

$$A - A^{\theta} = \cancel{R} + S - \cancel{R} + S$$

$$A - A^{\theta} = 2S$$

$$\frac{A - A^{\theta}}{2} = S = Q.$$

Hence $A = P + Q$ is unique

$\Rightarrow$ Every square matrix can be expressed in one and only one way as a sum of hermitian and skew hermitian matrix

Every square Matrix can be expressed in one and only one way as $P + iQ$ where P and Q are hermitian.

Do it yourself.

* Show that every skew hermitian matrix A can be uniquely expressed as $\underline{P - iQ}$

where P — real skew symmetric

Q — Real symmetric.

Proof A is skew hermitian matrix.

$$\therefore A^{\theta} = -A. \quad \text{—} (1)$$

$$A = \frac{A}{2} + \frac{A}{2} + \frac{\bar{A}}{2} - \frac{\bar{A}}{2}$$

$$= \frac{1}{2}(A + \bar{A}) - \frac{1 \times i}{2i}(-A + \bar{A})$$

$$A = P - iQ.$$

where $P = \frac{1}{2}(A + \bar{A})$ $Q = \frac{1}{2i}(\bar{A} - A)$

$$P = \frac{1}{2}(A + \bar{A})$$

$$P' = \frac{1}{2}(A + \bar{A})' = \frac{1}{2}\left[A' + (\bar{A})'\right]$$

$$= \frac{1}{2}\left[A' + A^{\theta}\right]$$

$$= \frac{1}{2}\left[(-A^{\theta})' + A^{\theta}\right]$$

$$= \frac{1}{2}\left[-((\bar{A})')' - A\right] \{\text{from } ①\}$$

$$= \frac{1}{2}\left[-\bar{A} - A\right] = -\frac{1}{2}(A + \bar{A})$$

$$P' = -P$$

$\therefore$ P is skew symmetric matrix. — (i)

$$P = \frac{1}{2}(A + \bar{A})$$

$$\bar{P} = \frac{1}{2}\overline{(A + \bar{A})} = \frac{1}{2}(\bar{A} + (\bar{\bar{A}}))$$

$$= \frac{1}{2}(\bar{A} + A) = \frac{1}{2}(A + \bar{A}) = P$$

$\bar{P} = P$ $\Rightarrow$ P is real — (ii) from (ii) and (iii)

P is real skew symmetric Matrix

$$Q = \frac{1}{2i}(\bar{A} - A)$$

$$Q' = \frac{1}{2i}(\bar{A} - A)' = \frac{1}{2i}[(\bar{A})' - A']$$

$$= \frac{1}{2i}\{A^{\theta} + (A^{\theta})'] \text{ (from (i))}$$

$$= \frac{1}{2i}\left[-A + \left(\left(\bar{A}\right)'\right)'\right]$$

$$= \frac{1}{2i}\left[-A + \bar{A}\right] = \frac{1}{2i}\left(\bar{A} - A\right) = Q.$$

$$Q' = Q.$$

Hence Q is symmetric Matrix. — (iv)

$$\bar{Q} = \overline{\left(\frac{1}{2i}\left(\bar{A} - A\right)\right)} = \frac{-1}{2i}\left(\overline{\bar{A} - A}\right)$$

$$= \frac{-1}{2i}\left(\left(\bar{\bar{A}}\right) - \bar{A}\right)$$

$$= \frac{-1}{2i}\left(A - \bar{A}\right)$$

$$= \frac{1}{2i}\left(\bar{A} - A\right) = Q.$$

$\bar{Q} = Q. \Rightarrow Q$ is real — (V)

from (IV) and (V).

Q is real symmetric Matrix

$\therefore$ A Can be expressed in form of $P - iQ$
where P is real skew symmetric and

Q is real symmetric Matrix

Now we have to prove $A = P - iQ$ is unique.

Let $A = R - iS$ — (VI) where R is real skew symmetric, S is real symmetric

$\bar{A} = \overline{R - iS}$

$= \bar{R} + i\bar{S}$

$\bar{R} = R$

$\bar{S} = S.$

$\bar{A} = R + iS.$ — (VII)

Add VI and VII

$$A + \bar{A} = 2R$$

$$\frac{A + \bar{A}}{2} = R. = P$$

subtract VI from VII

$$\frac{\bar{A} - A}{2i} = s. = Q.$$

Hence $A = R - is$ is same as $A = P - iQ$.

Hence representation is unique.

Express the Matrix $A = \begin{bmatrix} 1+i & 2i & 3 \\ 0 & 2-3i & 3-4i \\ 5 & -7i & 0 \end{bmatrix}$

as the sum of hermitian and skew hermitian matrix.

Sol

$$A = \begin{bmatrix} 1+i & 2i & 3 \\ 0 & 2-3i & 3-4i \\ 5 & -7i & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1+i & 0 & 5 \\ 2i & 2-3i & -7i \\ 3 & 3-4i & 0 \end{bmatrix}$$

$$A^{\theta} = \left(\overline{A'}\right) = \begin{bmatrix} 1-i & 0 & 5 \\ -2i & 2+3i & 7i \\ 3 & 3+4i & 0 \end{bmatrix}$$

$$A + A^{\theta} = \begin{bmatrix} 1+i & 2i & 3 \\ 0 & 2-3i & 3-4i \\ 5 & -7i & 0 \end{bmatrix} + \begin{bmatrix} 1-i & 0 & 5 \\ -2i & 2+3i & 7i \\ 3 & 3+4i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2i & 8 \\ -2i & 4 & 3+3i \\ 8 & 3-3i & 0 \end{bmatrix}$$

$$\frac{A+A^{\theta}}{2} = \begin{bmatrix} 1 & i & 4 \\ -i & 2 & 3/2+3/2\, i \\ 4 & 3/2-3/2\, i & 0 \end{bmatrix}$$ which is hermitian

$$A - A^{\theta} = \begin{bmatrix} 1+i & 2i & 3 \\ 0 & 2-3i & 3-4i \\ 5 & -7i & 0 \end{bmatrix} - \begin{bmatrix} 1-i & 0 & 5 \\ -2i & 2+3i & 7i \\ 3 & 3+4i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 2i & -2 \\ 2i & -6i & 3-11i \\ 2 & -3-11i & 0 \end{bmatrix}$$

$$\frac{A - A^{\theta}}{2} = \begin{bmatrix} i & i & -1 \\ i & -3i & 3/2 - 11/2\, i \\ 1 & -3/2 - 11/2\, i & 0 \end{bmatrix}$$

Which is skew hermitian Matrix

Hence $A = \left(\dfrac{A + A^{\theta}}{2}\right) + \left(\dfrac{A - A^{\theta}}{2}\right)$

$$= \begin{bmatrix} 1 & i & 4 \\ -i & 2 & 3/2 + 3/2\, i \\ 4 & 3/2 - 3/2\, i & 0 \end{bmatrix} + \begin{bmatrix} i & i & -1 \\ i & -3i & 3/2 - 11/2\, i \\ 1 & -3/2 - 11/2\, i & 0 \end{bmatrix}$$

$\therefore$ Matrix A has been expressed as the sum of hermitian and skew hermitian Matrix.