General Theorems: Important Questions

- Rolle's thm.
- Lagrange's thm.
- Cauchy's Mean Value thm.
- Taylor's thm.

use Cauchy's Mean Value thm. to evaluate $\lim_{x \to 1} \frac{\cos \pi x/2}{\log 1/x}$

Sol. $f(x) = \cos \frac{\pi x}{2}$ $g(x) = \log x$

$f(x)$ and $g(x)$ are Continous in $[x, 1]$

$f(x)$ and $g(x)$ are derivable in $(x, 1)$

also $g'(x) = \frac{1}{x} \neq 0$ in $(x, 1)$

$\therefore$ all Conditions of Cauchy's mean value thm are satisfied.

$$\Rightarrow \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$b = 1$
$a = x$

$$\Rightarrow \quad \frac{f(1) - f(x)}{g(1) - g(a)} = \frac{f'(c)}{g'(c)}$$

where $x < c < 1$

$$\Rightarrow \frac{\cos \pi/2 - \cos \frac{\pi x}{2}}{\log 1 - \log x} = \frac{-\pi/2 \sin \frac{\pi c}{2}}{1/c}$$

$$\Rightarrow \frac{-\cos \frac{\pi x}{2}}{\log 1/x} = \frac{-\pi/2 \sin \frac{\pi c}{2}}{1/c}$$

Taking Lim both side

$x < c < 1$

$x \to 1$

$c \to 1$

$$\Rightarrow \lim_{x\to 1} \frac{\cos\frac{\pi x}{2}}{\log 1/x} = \lim_{c\to 1} \frac{\pi/2 \ \sin\frac{\pi c}{2}}{1/c}$$

$$\Rightarrow \lim_{x\to 1} \frac{\cos \pi x/2}{\log 1/x} = \pi/2 \sin \pi/2 \qquad \{\sin \pi/2 = 1\}$$

$$\Rightarrow \lim_{x\to 1} \frac{\cos \pi x/2}{\log 1/x} = \frac{\pi}{2} \quad \underline{\underline{\text{Ans.}}}$$

② Using Cauchy's Mean Value theorem

Prove that for $x > 1$

$$\lim_{n \to \infty} n\left(x^{1/n} - 1\right) = \log x$$

Proof

$f(x) = x^{1/n}$ $\qquad$ $g(x) = \log x$

$f(x)$ and $g(x)$ are continous in $[1, x]$

$f(x)$ and $g(x)$ are derivable in $(1, x)$

$g'(x) = \frac{1}{x} \neq 0$ in $(1, x)$

$\therefore$ all Condition of Cauchy's mean Values thm satisfied

$$\frac{f(x)-f(1)}{g(x)-g(1)} = \frac{f'(c)}{g'(c)} \quad \text{where } 1 < c < x$$

$$\frac{x^{1/n}-1}{\log x - \log 1} = \frac{\frac{1}{n}c^{\frac{1}{n}-1}}{\frac{1}{c}}$$

$$\frac{x^{1/n}-1}{\log x} = \frac{1}{n}c^{\frac{1}{n}}$$

$$n\left(x^{1/n}-1\right) = c^{1/n}\cdot\log x$$

Take $\lim_{n\to\infty}$ both sides.

$$\lim_{n\to\infty} n\left(x^{1/n}-1\right) = \log x \lim_{n\to\infty} c^{1/n}$$

$$\lim_{n\to\infty} n\left(x^{1/n}-1\right) = \log x.$$

Hence Proved.