EIGEN VALUES AND CAYLEY-HAMILTON THEOREM

Important Question (PYQ)

Determine the eigen values and eigen vectors of a matrix.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Is it diagonalisable? Justify.

Sol:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Characteristic eq. of A.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3$$

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - R_1$$

$$(6-\lambda)(2-\lambda)^2 = 0$$

$$(6-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\lambda = 2, 2, 6.$$

$\therefore$ eigen values are $2, 2, 6.$

Let X is the eigen vector of A.

$$(A - \lambda I)X = 0$$

Now for $\lambda = 6$ eigen vector is

1. Characteristic eq.
2. eigen values.
3. spectrum of matrix.
4. eigen values + eigen vector

$$(A - 6I)X = 0$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$C_1 \leftrightarrow C_3.$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Rank of Coeff matrix is 2.

$\therefore$ eq. have only $3-2=1$ L.I. solution.

$$x + y - 3z = 0$$
$$-4y + 8z = 0$$

$$z = k.$$

$$-4y + 8k = 0$$
$$-4y = -8k$$
$$y = 2k$$
$$x + 2k - 3k = 0$$
$$x = k.$$

$$\therefore \text{ eigen vector is } \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\frac{x}{\begin{vmatrix} 1 & -3 \\ -4 & 8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -3 \\ 0 & 8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 0 & -4 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{-y}{8} = \frac{z}{4}.$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

eigen vector for $\lambda = 2$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now Coeff matrix is of Rank 1.

$\therefore$ eq. have $3-1=2$ solutions

$$x_1 + y_1 + z_1 = 0.$$

$y_1 = k \quad z_1 = 0$

$x_1 + k + 0 = 0$

$x_1 = -k \quad \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$y_1 = 0 \quad z_1 = k$

$x_1 + 0 + k = 0$

$x_1 = -k$

$\begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ eigen vectors for $\lambda = 2$ are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Yes this is diagonalisable.

✲ invertible Matrix.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

✲ P

Now find $P^{-1}AP$

Diagonalise the Matrix. A

$$|P| = 1(1) + 1(2) - 1(-1)$$

$$= 1 + 2 + 1 = 4.$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & -1 & 1 \end{matrix}$$

$$\text{Adj } P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 6 & -2 & -2 \\ 12 & 2 & 0 \\ 6 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$