CALCULUS

State and Prove Taylor's Theorem

Statement :- If a function f defined in $[a, b]$ s.t.

(i) $f, f', f'', f''', \ldots f^{n-1}(x)$ are continous in $[a, b]$

(ii) f^n exists in (a, b)

then $\exists$ at least one real no. $c \in (a, b)$ s.t.

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a)$$

$$+ \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(b-a)^n}{n!} f^n(c)$$

Sol. Consider a function ϕ s.t

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} f''(x) +$$

$$- - - - - - + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) +$$

$$\frac{(b-x)^n}{n!} A . \quad \text{--- (1)}$$

Where A is Constant to determine s.t

$$\phi(a) = \phi(b)$$

$$f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + - - - -$$

$$- - - + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} A = f(b) \quad \text{--- (2)}$$

$f, f'', f', f''' \_\_\_ f^{n-1}$ are

Continous in $[a, b]$

Sum of Continous function is Continous

$\therefore$ $\phi(x)$ is Continous in $[a, b]$ — ③

IIly $\phi(x)$ is derivable in (a, b) — ④

$\phi(a) = \phi(b)$ — ⑤

from ③, ④ and ⑤

By Rolle's thm.

$\exists \; c \in (a,b)$ s.t.

$\phi'(c) = 0 \quad — \quad ⑥$

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} f''(x) + \cdots + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(b-x)^n}{n!} A$$

$$\phi'(x) = f'(x) + \left[(b-x)f''(x) + f'(x)(-1)\right] +$$

$$\left[\frac{(b-x)^2}{2!} f'''(x) + f''(x) \cdot \frac{2(b-x)(-1)}{2!}\right]$$

$$+ \quad - - - \cdot + \left[\frac{(b-x)^{n-1}}{(n-1)!} f^{n}(x) + f^{n-1}(x) \cdot (n-1) \frac{(b-x)^{n-2}(-1)}{(n-1)!}\right.$$

$$+ \quad \frac{n(b-x)^{n-1}}{n!}(-1) A$$

$$= \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(b-x)^{n-1}}{(n-1)!} A$$

$$\phi'(x) = \frac{(b-x)^{n-1}}{(n-1)!} \left(f^n(x) - A \right)$$

from 6

$$\phi'(c) = 0$$

$$\frac{(b-c)^{n-1}}{(n-1)!}\left[f^{n}(c)-A\right]=0$$

$$f^{n}(c)-A=0$$

$$\underbrace{f^{n}(c)=A}.$$

Put $f^{n}(c)=A$ in ②

$$f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+ \cdots$$

$$- \cdots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c) = f(b)$$

Hence Proved.