Leibnitz's Theorem: Example

Most Important

Prove that

$$\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right]$$

Given that $x > 0$.

$$y = \frac{\log x}{x} = \log x \cdot \frac{1}{x}$$

$$\quad\quad\quad\quad\quad v \quad\quad u$$

$$u = \frac{1}{x} = x^{-1}$$

$$u_1 = (-1)\, x^{-1-1} = (-1)\, x^{-2}$$

$$u_2 = (-1)(-2)\, x^{-3} = (-1)^2\, 2!/x^3$$

$$u_3 = (-1)(-2)(-3)\, x^{-4} = (-1)^3\, 3!/x^4$$

$\vdots$

$$u_n = \frac{(-1)^n n!}{x^{n+1}} \quad \text{--- (i)}$$

$$v = \log x$$

$$v_1 = \frac{1}{x} = x^{-1}$$

$$v_2 = (-1)x^{-2}$$

$$u_3 = (-1)(-2)x^{-3} = \frac{(-1)^2\, 2!}{x^3}$$

$$u_4 = (-1)(-2)(-3)x^{-4} = \frac{(-1)^3\, 3!}{x^4}$$

$$\vdots$$

$$u_n = \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \text{--- (II)}$$

$$y = \frac{\log x}{x}$$

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots\ldots\ldots + {}^nC_r\, u_{n-r} v_r + \ldots\ldots\ldots + {}^nC_n\, u\, v_n$$

$$= \underline{\underline{u\,v}}$$

$$y_n = (uv)_n = {}^nC_0 \frac{(-1)^n n!}{x^{n+1}} \cdot \log x + {}^nC_1 \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x} + {}^nC_2 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot \left(\frac{-1}{x^2}\right)$$

$$+ \text{-----} + {}^nC_n \frac{1}{x} \cdot \frac{(-1)^{n-1}(n-1)!}{x^n}$$

[from (i) and (ii)]

$$= \frac{(-1)^n n!}{x^{n+1}} \log x + n \frac{(-1)^{n-1}(n-1)!}{x^{n+1}} +$$

$$\frac{n(n-1)}{2} \frac{(-1)^{n-1}(n-2)!}{x^{n+1}} + \text{----}$$

$$\frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right]$$

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right] = \underline{\underline{R.H.S}}$$

Hence Proved.