Leibnitz's Theorem: State And Proof

If u and v are functions of x possessing n^{th} order derivatives then

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_{r-1}\, u_{n-r+1} v_{r-1} + {}^nC_r\, u_{n-r} v_r + \ldots + {}^nC_n\, u v_n$$

Proof :- $(uv)_1 = u v_1 + v u_1$

$$(uv)_1 = {}^1C_0\, u_1 v + {}^1C_1\, u v_1$$

$$= u_1 v + v_1 u.$$

Result is true for $n=1$.

Assume that the theorem is true for $n = m$.

$$\therefore\ (uv)_m = {}^mC_0\, \underset{1}{u_m} \underset{2}{v} + {}^mC_1\, \underset{1}{u_{m-1}} \underset{2}{v_1} +$$

$$ {}^mC_2\, \underset{1}{u_{m-2}} \underset{2}{v_2} + \text{-----}$$

$$+ {}^mC_{r-1}\, \underset{1}{u_{m-r+1}} \underset{2}{v_{r-1}} +$$

$$ {}^mC_r\, \underset{1}{u_{m-r}} \underset{2}{v_r} + \text{-----}$$

$$+ {}^mC_m\, \underset{1}{u} \underset{2}{v_m}.$$

Differentiate Both sides.

$$(uv)_{m+1} = {}^mC_0 \left[u_m v_1 + v\, u_{m+1} \right] +$$

$$ {}^{m}C_{1} \left[u_{m-1} v_{2} + v_{1} u_{m} \right] + $$

$$ {}^{m}C_{2} \left[u_{m-2} v_{3} + v_{2} u_{m-1} \right] + $$

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$$ \text{— — — — - - .} + $$

$$ {}^{m}C_{r-1} \left[u_{m-r+1} v_{r} + v_{r-1} u_{m-r+2} \right] + $$

$$ {}^{m}C_{r} \left[u_{m-r} v_{r+1} + v_{r} u_{m-r+1} \right] + $$

$$ \text{— - — — — — - -} + $$

$$ {}^{m}C_{m} \left[u\, v_{m+1} + v_{m} u_{1} \right] $$

$$= {}^mC_0 \, u_{m+1} v + u_m v_1 \left[{}^mC_0 + {}^mC_1\right] +$$

$$u_{m-1} v_2 \left[{}^mC_1 + {}^mC_2\right] + \text{-----}$$

$$\text{-----} + u_{m-r+1} v_r \left[{}^mC_{r-1} + {}^mC_r\right]$$

$$+ \text{-----} + {}^mC_m \, u \, v_{m+1}$$

$${}^mC_0 = {}^{m+1}C_0 = 1 \qquad {}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1 \qquad {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

$${}^mC_m = {}^{m+1}C_{m+1}$$

$$(uv)_{m+1} = {}^{m+1}C_0\, u_{m+1} v + {}^{m+1}C_1\, u_m v_1 + {}^{m+1}C_2\, u_{m-1} v_2 + {}^{m+1}C_r\, u_{m-r+1} v_r + \text{-------} + {}^{m+1}C_{m+1}\, u v_{m+1}$$

$\therefore$ theorem is true for $n = \underline{\underline{m+1}}$

$\therefore$ By Induction Method.

Result is true for all +ve integers.

Hence Proved.