Leibnitz's Theorem: Example

Most Important

$$y = (\sinh^{-1} x)^2$$

Prove that

$$(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2 y_n = 0$$

Sol

$$y = (\sinh^{-1} x)^2$$

$$y_1 = 2(\sinh^{-1} x) \frac{d}{dx} \sinh^{-1} x$$

$$= 2\sinh^{-1}x \cdot \frac{1}{\sqrt{x^2+1}}$$

$$\sqrt{x^2+1}\, y_1 = 2\sinh^{-1}x$$

squaring both side

$$(x^2+1)\, y_1^2 = 4\left(\sinh^{-1}x\right)^2$$

$$\underbrace{(x^2+1)}_{1}\, \underbrace{y_1^2}_{2} = 4y$$

Differentiate Both sides

$$(x^2+1)\,2y_1y_2 + y_1^2(2x) = 4y_1$$

$$\cancel{2}y_1\;(x^2+1)y_2 + xy_1 = \cancel{4}^{\,2}y_1$$

$$(x^2+1)y_2 + xy_1 = 2$$

$$(x^2+1)y_2 + xy_1 - 2 = 0$$

Diff. n times Both sides

$$\left((x^2+1)y_2\right)_n + (xy_1)_n - 0 = 0 \quad \text{--- (1)}$$

$$\big((x^2+1)\,y_2\big)_n$$

$$\underset{v}{(x^2+1)} \qquad \underset{u}{y_2}$$

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \dots\dots\dots + {}^nC_r\, u_{n-r}v_r + \dots\dots\dots + {}^nC_n\, u\, v_n$$

$$= {}^nC_0\, y_{n+2}\,(x^2+1) + {}^nC_1\, y_{n+1}\,(2x) + {}^nC_2\, y_n \cdot 2$$

$$= y_{n+2}\,(x^2+1) + n\, y_{n+1}\,(2x) + \frac{n(n-1)}{\cancel{2}} \cdot y_n \cdot \cancel{2}$$

$$= (x^2+1)\,y_{n+2} + \underline{2nx}\ \underline{y_{n+1}} + (n^2-n)\,y_n \quad \text{——} \ (11)$$

$$(\underset{\downarrow\atop v}{x}\ \underset{\downarrow\atop u}{y_1})_n$$

$$\boxed{(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \dots\dots\dots + {}^nC_r\, u_{n-r}v_r + \dots\dots\dots + {}^nC_n\, u\, v_n}$$

$$= {}^nC_0\, y_{n+1}\, x + {}^nC_1\, y_n \cdot (1)$$

$$= x\, y_{n+1} + n\, y_n \quad \text{— (iii)}$$

Put values of (ii) and (iii) in (i)

$$(x^2+1)y_{n+2} + 2nx\,y_{n+1} + (n^2-n)y_n$$
$$+ x\,y_{n+1} + ny_n = 0$$

$$(x^2+1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-n+n)y_n = 0$$

$$(x^2+1)y_{n+2} + (2n+1)x\,y_{n+1} + n^2 y_n = 0$$

Hence Proved