Leibnitz's Theorem: Example

Most Important

$y = (\sin^{-1} x)^2$ find $y_n(0)$

Sol

$$y = (\sin^{-1} x)^2$$

$$y_1 = 2 \sin^{-1} x \frac{d}{dx} \sin^{-1} x$$

$$= 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = 2 \sin^{-1}x$$

Squaring both sides

$$(1-x^2)(y_1)^2 = 4(\sin^{-1}x)^2$$

$$(1-x^2)\, y_1^2 = 4y.$$

$$(1-x^2)\, y_1^2 - 4y = 0$$

Diff again.

$$(1-x^2)\, 2y_1 \cdot y_2 + y_1^2 \cdot (-2x) - 4y_1 = 0$$

$$2y_1\left[(1-x^2)y_2+(-xy_1)-2\right]=0$$

$$(1-x^2)y_2-xy_1-2=0 \quad \text{— } A$$

Diff n times Both sides

$$\left((1-x^2)y_2\right)_n-(xy_1)_n-(2)_n=0$$

$$\underline{\left((1-x^2)y_2\right)_n}-\underline{(xy_1)_n}=0 \quad \text{— } ①$$

$$\Big(\underbrace{(1-x^2)}_{v}\,\underbrace{y_2}_{u}\Big)_n$$

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1}v_1 + {}^nC_2\, u_{n-2}v_2 + \dots\dots\dots + {}^nC_r\, u_{n-r}v_r + \dots\dots\dots + {}^nC_n\, u\, v_n$$

$$= {}^nC_0\, y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)$$

$$= (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$$

$$= (1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n \quad \text{—(i)}$$

$(x y_1)_n$

$\downarrow \quad \downarrow$
$v \quad u$

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \dots\dots\dots + {}^nC_r\, u_{n-r} v_r + \dots\dots\dots + {}^nC_n\, u\, v_n$$

$= {}^nC_0\, y_{n+1}\, x + {}^nC_1\, y_n (1)$

$= x y_{n+1} + n y_n$ —— (III)

Put (II) and (III) in (I)

$(1 - x^2) y_{n+2} - 2nx\, y_{n+1} - n(n-1) y_n - x\, y_{n+1} - n y_n = 0$

$$(1-x^2)y_{n+2} - (2n+1)y_{n+1}x - (n^2-n+n)y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)y_{n+1}x - n^2 y_n = 0 \quad \text{— IV}$$

y at $x=0$

$y = 0$ — (i)

$y_1 = 0$ — (ii)

from * $y_2 = 2$ — (iii)

Now from IV $y_{n+2} = n^2 y_n(0)$ —(iv) from(iv) $y_3 = 1(0)$ — from (ii)

$y_3 = 0$

$y_4 = 2^2 y_2(0) = 2^2 \cdot 2$ — from(iii)

$y_5 = 3^2 y_3(0) = 0$

$y_6 = 4^2 y_4(0) = 4^2 \cdot 2^2 \cdot 2$

$\vdots$

$$y_n(0) = \begin{cases} 0 & n \text{ is odd} \\ 2 \cdot 2^2 \cdot 4^2 \cdots (n-2)^2 & \text{when } n \text{ is even and } n \neq 2. \end{cases}$$

Ans

Like