Leibnitz's Theorem: Example

find the n^{th} derivative of

(i) $y = e^x \sin x$

(ii) $y = x^2 e^x \cos x$

$y = \underset{u}{e^x} \; \underset{v}{\sin x}$ $\qquad$ $y_n = ?$

$$u = e^x \qquad v = \sin x$$

$$u_1 = e^x \qquad v_1 = \cos x = \sin\left(\frac{\pi}{2} + x\right)$$

$$u_2 = e^x \qquad v_2 = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right)$$

$$= \sin\left(\frac{2\pi}{2} + x\right)$$

$$u_3 = e^x \qquad v_3 = \cos\left(\frac{2\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} + \frac{2\pi}{2} + x\right)$$

$$= \sin\left(3\pi/2 + x\right)$$

$$\vdots$$

$$u_n = e^x$$

$$v_n = \sin\left(\frac{n\pi}{2} + x\right)$$

$$y_n = (uv)_n =$$

2.2 LEIBNITZ'S THEOREM

Let u and v be functions of x. Then Leibnitz's theorem states that

$$\frac{d^n}{dx^n}(uv) = (uv)_n = {}^nC_0\, u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n\, u v_n \tag{2.1}$$

$${}^nC_0\, e^x \sin x +$$

$${}^nC_1\, e^x \sin\left(\frac{\pi}{2} + x\right) + {}^nC_2\, e^x \sin\left(\frac{2\pi}{2} + x\right) + \cdots$$

$$\cdots + {}^nC_n\, e^x \sin\left(\frac{n\pi}{2} + x\right)$$

$$e^x \left[\sin x + n \sin\left(\frac{\pi}{2} + x\right) + \frac{n(n-1)}{2} \sin\left(\frac{2\pi}{2} + x\right) \right.$$

$$\left. + \; \text{---} \; \text{-} \; \text{-} \; \text{-} \; \text{-} \; + \sin\left(\frac{n\pi}{2} + x\right) \right] \quad \underline{\underline{\text{Ans}}}$$

$$y = x^2 e^x \cos x$$

$$= \underbrace{e^x \cos x}_{u} \cdot \underbrace{x^2}_{v}$$

$u = e^{x} \cos x$

$v = x^2$

$v_1 = 2x$

$v_2 = 2$

$v_3 = 0$

$\vdots \quad 0$

$$\boxed{\begin{aligned} u &= e^{ax}\cos(bx+c) \\ u_n &= (a^2+b^2)^{n/2} e^{ax} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right) \end{aligned}}$$

$$u_n = (1+1)^{n/2} e^{x} \cos(x + n\tan^{-1} 1)$$

$$= 2^{n/2} e^{x} \cos\left(x + \frac{n\pi}{4}\right)$$

$y_n = (uv)_n$

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$$= {}^nC_0 \left[2^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right) \right] x^2 +$$

$${}^nC_1 \left[2^{\frac{n-1}{2}} e^x \cos\left(x + \frac{(n-1)\pi}{4}\right) \right] 2x +$$

$${}^nC_2 \left[2^{\frac{n-2}{2}} e^x \cos\left(x + \frac{(n-2)\pi}{4}\right) \right] 2$$

$\frac{n-1}{2} + 1$

$\frac{n-1+2}{2}$

$\frac{n+1}{2}$

$\frac{n+1}{2} - \frac{n-2}{2}$

$\frac{n+1-n+2}{2}$

$3/2$

$$e^{x}\, 2^{\frac{n-2}{2}} \left[2 \cos\left(x + \frac{n\pi}{4}\right) \cdot x^{2} + 2^{3/4} \cdot x \cdot \cos\left(x + \frac{(n-1)\pi}{4}\right) \right.$$

$$\left. + \frac{n(n-1)}{2} \left(2 \cos\left(x + \frac{(n-2)\pi}{4}\right)\right) \right]$$

Ans.