CALCULUS

State and Prove Lagrange's Mean Value Theorem

Statement :- If a function f is

(i) Continous in $[a,b]$

(ii) differentiable in (a,b)

then $\exists$ at least one real no. $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Consider a function

$$\phi(x) = f(x) + Ax \quad \text{--- (1)}$$

where A is constant to determine

s.t. $\phi(a) = \phi(b)$

from (1) $f(a) + A \cdot a = f(b) + A \cdot b$

$$A \cdot (a - b) = f(b) - f(a)$$

$$-A\,(b - a) = f(b) - f(a)$$

$$-A = \frac{f(b) - f(a)}{b - a} \quad \text{--- (2)}$$

(i) f is Continous in $[a,b]$ [Given]

Ax is Continous in $[a,b]$ [$\because$ Ax is a Polynomial]

also sum of Continous functions is Continous.

$\therefore$ $\phi(x)$ is Continous in $[a,b]$ {from (1)] — (3)

(ii) f is differentiable in (a,b) (Given)

Ax is differentiable in (a,b) (Ax is Polynomial)

Sum of differentiable functions is differentiable.

$\therefore$ $\phi(x)$ is differentiable in (a,b) [from ①] — ④

Now

$\phi(a) = \phi(b)$ — ⑤

$\therefore$ By Rolle's thm. {from ③, ④ & ⑤}

$\phi(x) = f(x) + Ax$

$\phi'(x) = f'(x) + A$

$\phi'(c) = 0$

$f'(c) + A = 0$

$$f'(c) = -A$$

$$= \frac{f(b) - f(a)}{b - a} \quad \{\text{from } ②\}$$

Hence $f'(c) = \dfrac{f(b) - f(a)}{b - a}.$

Hence Proved.