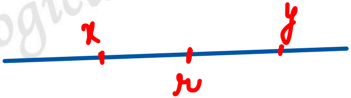
Density Theorem : Proof

Between any two distinct real numbers there is always a rational number.

(Or)



$x, y \in \mathbb{R}$, $x < y$ $\exists$ $r \in \mathbb{Q}$

s.t. $x < r < y$

Proof

x, y are two distinct real No.

s.t. $x < y$ & $x > 0$

$y - x > 0$

$n(y-x) > 1$

$ny - nx > 1$

$ny > 1 + nx$

$1 + nx < ny.$ — ①

$nx \in \mathbb{R}.$

$\Rightarrow \exists$ a unique integer m

s.t. $m \leq nx < m+1$ — ②

Archimedean

$a > 0 \quad b \in \mathbb{R}$

$\exists$ a natural no. n s.t.

$na > b$

$a = y - x$

$b = 1$

$x \in \mathbb{R}$ $\exists$ a unique integer m s.t. $m \leq x < m+1$

$m \leq nx$ from 2.

$m+1 \leq nx+1 < ny$ (from 1)

$m+1 < ny$ — 3

$nx < m+1$ — 4 (from 2)

from 3 and 4

$nx < m+1 < ny$

$x < \frac{m+1}{n} < y$

Now $\frac{m+1}{n}$ is a rational no. b/w x & y.

Hence Between any two distinct real numbers $\exists$ a rational No.

Hence Proved.