CALCULUS

State and Prove Cauchy's Mean Value Theorem

Statement: f and g are two functions s.t.

(i) f and g are continous in $[a, b]$

(ii) Both are differentiable in (a, b)

(iii) $g'(x) \neq 0$ for any $x \in (a, b)$

then $\exists$ at least one real no $c \in (a, b)$

s.t. $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

Proof

Let $g(b) = g(a)$

$\left.\begin{array}{l} g \text{ is Continous in } [a,b] \\ g \text{ is differentiable in } (a,b) \end{array}\right]$ Given.

$\therefore$ By Rolle's thm.

$g'(c) = 0 \qquad c \in (a,b)$

But $g'(x) \neq 0 \quad x \in (a,b)$ {Given}

Which is contradiction.

Hence $g(a) \neq g(b)$

$$\Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} \text{ is meaningful.}$$

Now Let's Consider a function

$$\phi(x) = f(x) + A\, g(x) \quad \text{---} ①$$

Where A is Constant to determine

$$\phi(a) = \phi(b)$$

$$f(a) + A\, g(a) = f(b) + A\, g(b)$$

$$A\{g(a) - g(b)\} = f(b) - f(a)$$

$$-A\,(g(b) - g(a)) = f(b) - f(a)$$

$$-A = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{--- } 2$$

f is Continous in $[a, b]$
g is Continous in $[a, b]$ } Given

sum of Continous functions is Continous

So from ①

$\phi(x)$ is Continous in $[a, b]$

also f and g are differentiable in (a, b) — ③ (Given)

$\therefore$ $\phi(x)$ is differentiable in (a, b) — ④ from ① (sum of differentiable functions)

$\phi(a) = \phi(b)$ — ⑤

from ③, ④ and ⑤

By Rolle's thm.

$\phi'(c) = 0$ — ⑥ where $c \in (a, b)$

$$\phi(x) = f(x) + A\,g(x)$$

$$\phi'(x) = f'(x) + A\,g'(x) \quad \text{— (7)}$$

$$f'(c) + A\,g'(c) = 0 \quad (\text{from } (6) \text{ \& } (7))$$

$$f'(c) = -A\,g'(c)$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (\text{from } (2))$$

$c \in (a, b)$

Hence Proved.