$$p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$$

Prove that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

Sol $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$$p^2 = a^2\left(\frac{1+\cos 2\theta}{2}\right) + b^2\left(\frac{1-\cos 2\theta}{2}\right)$$

$$2\rho^2 = a^2 + a^2\cos 2\theta + b^2 - b^2\cos 2\theta$$

$$= (a^2 + b^2) + (a^2 - b^2)\cos 2\theta$$

$$2\rho^2 - (a^2 + b^2) = (a^2 - b^2)\cos 2\theta \quad -(1)$$

Differentiate both sides

$$2\left(2\rho\frac{d\rho}{d\theta}\right) = (a^2 - b^2)(-\sin 2\theta)(2)$$

$$2\rho \frac{d\rho}{d\theta} = -(a^2 - b^2)\sin 2\theta \quad \text{— (II)}$$

Squaring and adding (I) and (II)

$$4\rho^4 + (a^2+b^2)^2 - 2\cdot 2\rho^2(a^2+b^2) + 4\rho^2\left(\frac{d\rho}{d\theta}\right)^2$$

$$= (a^2-b^2)^2\cos^2 2\theta + (a^2-b^2)^2\sin^2 2\theta$$

$$4\rho^4 + (a^2+b^2)^2 - 4\rho^2(a^2+b^2) + 4\rho^2\left(\frac{d\rho}{d\theta}\right)^2$$

$$= (a^2-b^2)^2(\cos^2 2\theta + \sin^2 2\theta)$$

$$4\rho^4 + (a^2+b^2)^2 - 4\rho^2(a^2+b^2) + 4\rho^2\left(\frac{d\rho}{d\theta}\right)^2 - (a^2-b^2)^2 = 0$$

$$\Rightarrow 4\rho^4 + 4\rho^2\left(\frac{d\rho}{d\theta}\right)^2 - 4\rho^2(a^2+b^2) + a^4 + b^4 + 2a^2b^2 - a^4 - b^4 + 2a^2b^2 = 0$$

$$\Rightarrow 4\rho^4 + 4\rho^2\left(\frac{d\rho}{d\theta}\right)^2 - 4\rho^2(a^2+b^2) + 4a^2b^2 = 0$$

$$\Rightarrow \quad 4\rho^2 \left(\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 - (a^2+b^2) + \frac{a^2b^2}{\rho^2} \right) = 0$$

$$\Rightarrow \quad \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 - (a^2+b^2) + (a^2b^2)(\rho^{-2}) = 0$$

Differentiate Both side

$$\Rightarrow \quad 2\rho \frac{d\rho}{d\theta} + 2\left(\frac{d\rho}{d\theta}\right)^{2-1} \frac{d}{d\theta} \cdot \frac{d\rho}{d\theta} + a^2b^2(-2)\rho^{-2-1} \frac{d\rho}{d\theta} = 0$$

$$\Rightarrow \quad 2\rho \frac{d\rho}{d\theta} + 2\frac{d\rho}{d\theta} \cdot \frac{d^2\rho}{d\theta^2} - \frac{2a^2b^2}{\rho^3}\frac{d\rho}{d\theta} = 0$$

$$\Rightarrow \quad 2\frac{d\rho}{d\theta}\left\{\rho + \frac{d^2\rho}{d\theta^2} - \frac{a^2b^2}{\rho^3}\right\} = 0$$

$$\Rightarrow \quad \rho + \frac{d^2\rho}{d\theta^2} - \frac{a^2b^2}{\rho^3} = 0$$

$$\Rightarrow \quad p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Hence proved.