Show that the roots of the equation $(1+x)^{2n}+(1-x)^{2n}=0$ are given by

$$\pm i \tan \frac{(2r-1)\pi}{4n} \quad \text{where } r = 0, 1, 2 \ldots n$$

Sol.

$$(1+x)^{2n} + (1-x)^{2n} = 0$$

$$(1+x)^{2n} = -(1-x)^{2n}$$

$$\left(\frac{1+x}{1-x}\right)^{2n} = (-1)$$

$$\frac{1+x}{1-x} = (-1)^{1/2n}$$

$$\frac{1+x}{1-x} = (\cos\pi + i\sin\pi)^{1/2n}$$

$$= \left\{\cos(2r\pi+\pi) + i\sin(2r\pi+\pi)\right\}^{1/2n}$$

$$= \cos\frac{(2r+1)\pi}{2n} + i\sin\frac{(2r+1)\pi}{2n}$$

$$\frac{1+x}{1-x} = \cos\theta + i\sin\theta \qquad \theta = \frac{(2r+1)\pi}{2n}$$

By Componendo and dividendo

$$\frac{1+x+(1-x)}{1+x-(1-x)} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$\frac{\cancel{2}}{\cancel{2}x} = \frac{(1+\cos\theta) + i\sin\theta}{-1(1-\cos\theta) + i\sin\theta}$$

$1 - \cos 2\theta$
$= 2\sin^2\theta$

$$= \frac{2\cos^2\theta/2 + i\,2\sin\theta/2\,\cos\theta/2}{-1(2\sin^2\theta/2) + i\,2\sin\theta/2\,\cos\theta/2}$$

$1 + \cos 2\theta$
$= 2\cos^2\theta$

$\sin 2\theta$
$2\sin\theta\cos\theta$

$$= \frac{2\cos\theta/2\left[\cos\theta/2 + i\sin\theta/2\right]}{2i\sin\theta/2\left[-\frac{1}{i}\sin\frac{\theta}{2} + \cos\theta/2\right]}$$

$$= \frac{1}{i}\cot\theta/2\,\frac{\cancel{\left\{\cos\theta/2 + i\sin\theta/2\right]}}{\cancel{\left[\cos\theta/2 + i\sin\theta/2\right)}}$$

$$\frac{1}{x} = \frac{1}{i}\cot\frac{\theta}{2}$$

$$x = i \tan \frac{\theta}{2}$$

$$= i \tan \frac{(2r+1)\pi}{2n} \times \frac{1}{2}$$

$$= i \tan \frac{(2r+1)\pi}{4n} \qquad r = 0, 1, 2 \, \ldots \, (2n-1)$$

$$x = \pm i \tan \frac{(2r-1)\pi}{4n} \qquad r = [0, 1, 2 \, \ldots \, n.]$$

Ans

Hence Proved.