Prove that the roots of the equation $(x-1)^n = x^n$ are

$$\frac{1}{2}\left[1 + i \cot \frac{r\pi}{2}\right]$$ where $r = 0, 1, 2 \ldots (n-1)$

Sol

$$(x-1)^n = x^n$$

$$\left(\frac{(x-1)}{x}\right)^n = 1$$

$$\frac{x-1}{x} = (1)^{1/n}$$

$$= (\cos 0 + i \sin 0)^{1/n}$$

$$\frac{x-1}{x} = [\cos(2r\pi + 0) + i \sin(2r\pi + 0)]^{1/n}$$

where $r = 0, 1, 2 \ldots (n-1)$

$$\frac{x}{x} - \frac{1}{x} = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$$

$$1 - \frac{1}{x} = \cos \theta + i \sin \theta \quad \left[\theta = \frac{2r\pi}{n}\right]$$

$$\frac{-1}{x} = -1 + \cos\theta + i\sin\theta$$

$$\frac{1}{x} = 1 - \cos\theta - i\sin\theta$$

$$x = \frac{1}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1}{2\sin^2\theta/2 - i\{2\sin\theta/2\cos\theta/2\}}$$

$1-\cos 2\theta$

$= 2\sin^2\theta$

$\sin 2\theta$

$= 2\sin\theta\cos\theta$

$$= \frac{1}{2i\sin\theta/2\left[\frac{1}{i}\sin\theta/2 - \cos\theta/2\right]}$$

$$= \frac{1}{2i\sin\theta/2\left[-i\sin\theta/2 - \cos\theta/2\right]}$$

$$= \frac{1}{-2i\sin\theta/2\left(\cos\theta/2 + i\sin\theta/2\right)}$$

$$= \frac{i}{-2i^2 \sin\theta/2}\left(\cos\theta/2 + i\sin\theta/2\right)^{-1}$$

$$= \frac{i}{2\sin\theta/2}\left(\cos\theta/2 - i\sin\theta/2\right)$$

$$= \frac{1}{2}\left[i\cot\frac{\theta}{2} + 1\right]$$

$$= \frac{1}{2}\left[i\cot\frac{2r\pi}{n}\times\frac{1}{2} + 1\right]$$

$\cos(-\theta)$

$= \cos\theta$

$\sin(-\theta)$

$= -\sin\theta.$

$$= \frac{1}{2}\left[i \cot \frac{r\pi}{n} + i \right] \text{ where } r = 0, 1, 2 \ldots (n-1)$$

Hence Proved