Primitive Roots of Unity

Show that each primitive 8^{th} root of unity satisfies $x^4+1=0$

Sol.

Let $r_1 = \text{Cis}\,\frac{2\pi}{8}$

$$r_1 = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}$$

$$r_1^8 = \left(\cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}\right)^8$$

$= (\cos 2\pi + i \sin 2\pi) = 1 + i(0) = 1$

$r_4^8 = 1$

$\therefore$ for +ve Integer $K < 8$ r_4^k is primitive 8^{th} root of unity iff $(K, 8) = 1$

$\therefore$ $K = 1, 3, 5, 7$

$\therefore$ r_4, r_4^3, r_4^5, r_4^7 are primitive 8^{th} roots of unity.

$$z_4 = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$z_4^3 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$z_4^5 = z_4^8 \cdot z_4^{-3} = (1)\, z_4^{-3} = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{-1}$$

$$\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$z_4^7 = z_4^8 \cdot z_4^{-1} = \cos\pi/4 - i\sin\pi/4$$

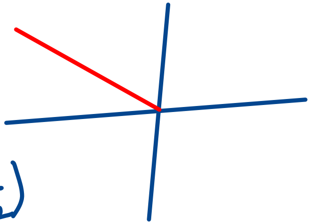
$$\boxed{\begin{aligned}&\cos(-\theta) = \cos\theta\\ &\sin(-\theta) = -\sin\theta\end{aligned}}$$

$$r_4^1 + r_4^7 = \cos\frac{\pi}{4} + \cancel{i\sin\frac{\pi}{4}} + \cos\frac{\pi}{4} - \cancel{i\sin\frac{\pi}{4}}$$

$$= 2\cos\frac{\pi}{4} = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}$$

$$r_4^5 + r_4^3 = \cos\frac{3\pi}{4} - \cancel{i\sin\frac{3\pi}{4}} + \cos\frac{3\pi}{4} + \cancel{i\sin\frac{3\pi}{4}}$$

$$= 2\cos\frac{3\pi}{4}$$



$$= 2\left(-\cos\frac{\pi}{4}\right) = 2\cdot\left(\frac{-1}{\sqrt{2}}\right)$$

$$= -\sqrt{2}$$

$$r_4 , r_4^3 , r_4^5 , r_4^7$$

$$(x - r_4)(x - r_4^3)(x - r_4^5)(x - r_4^7)$$

$$[(x - r_4)(x - r_4^7)][(x - r_4^3)(x - r_4^5)]$$

$$[x^2 - r_4^7 x - r_4 x + r_4^1 r_4^7][x^2 - r_4^5 x - r_4^3 x + r_4^3 r_4^5]$$

$$\{x^2 - x(r_4^7 + r_4^1) + 1\}\{x^2 - x(r_4^5 + r_4^3) + 1\}$$

$$\{x^2 - \sqrt{2}x + 1\}\{x^2 + \sqrt{2}x + 1\}$$

$$\{(x^2+1) - \sqrt{2}\,x\}\{(x^2+1) + \sqrt{2}\,x\}$$

$$(x^2+1)^2 - (\sqrt{2}\,x)^2 = x^4 + 1 + \cancel{2x^2} - \cancel{2x^2}$$

$$= \underline{x^4+1}$$

Equation with roots r_4, r_4^3, r_4^5, r_4^7 is

$$\underline{x^4+1}$$

Hence Proved.