Trigonometry And Matrices : De Moivre's Theorem

1. $$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$$

2. $$\frac{z^{2n} - 1}{z^{2n} + 1} = i\tan n\theta$$ $(z = \cos\theta + i\sin\theta)$

3. $$\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$$

Sol. ①

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

$$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$$

L.H.S

$$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$$

$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^4} = \frac{\cos 4\theta + i\sin 4\theta}{\left(\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)\right)^4}$$

$$= \frac{\cos 4\theta + i \sin 4\theta}{\cos(2\pi - 4\theta) + i \sin(2\pi - 4\theta)}$$

$$= \frac{\cos 4\theta + i \sin 4\theta}{\cos 4\theta - i \sin 4\theta}$$

$$= (\cos 4\theta + i \sin 4\theta)(\cos 4\theta - i \sin 4\theta)^{-1}$$

$$= (\cos 4\theta + i \sin 4\theta)(\cos 4\theta + i \sin 4\theta)$$

$\left[\because \cos(-\theta) = \cos\theta \right.$

$\sin(-\theta) = -\sin\theta$

$$= (\cos 4\theta + i \sin 4\theta)^2$$

$$= \cos 8\theta + i \sin 8\theta = R.H.S$$

Hence Proved.

(2)

$$Z = \cos\theta + i \sin\theta.$$

L.HS

$$\frac{Z^{2n} - 1}{Z^{2n} + 1} = \frac{(\cos\theta + i \sin\theta)^{2n} - 1}{(\cos\theta + i \sin\theta)^{2n} + 1}$$

$$= \frac{\cos 2n\theta + i \sin 2n\theta - 1}{\cos 2n\theta + i \sin 2n\theta + 1}$$

$$= \frac{-1\,(1 - \cos 2n\theta) + i \sin 2n\theta}{1 + \cos 2n\theta + i \sin 2n\theta}$$

$$= \frac{-2 \sin^2 n\theta + i \cdot 2 \sin n\theta \cos n\theta}{2 \cos^2 n\theta + i\,(2 \sin n\theta \cos n\theta)}$$

$$= \frac{2 \sin n\theta \, i \left(-\frac{\sin n\theta}{i} + \cos n\theta \right)}{2 \cos n\theta \left\{ \cos n\theta + i \sin n\theta \right\}}$$

$$= \frac{\cancel{2} \sin n\theta \, i \left(\frac{i^2 \sin n\theta}{i} + \cos n\theta \right)}{\cancel{2} \cos n\theta \left(\cos n\theta + i \sin n\theta \right)}$$

$$= \frac{i \tan n\theta \, \cancel{(\cos n\theta + i \sin n\theta)}}{\cancel{(\cos n\theta + i \sin n\theta)}} = i \tan n\theta = \underline{\underline{R.H.S}}$$

3. L.H.S $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$

$$= \left(\frac{\sin^2\theta+\cos^2\theta+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$$

$$= \left(\frac{\sin^2\theta-i^2\cos^2\theta+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$$

$$= \left(\frac{(\sin\theta + i\cos\theta)(\sin\theta - i\cos\theta) + (\sin\theta + i\cos\theta)}{1 + \sin\theta - i\cos\theta} \right)^n$$

$$= \left(\frac{(\sin\theta + i\cos\theta)\cancel{(\sin\theta - i\cos\theta + 1)}}{\cancel{1 + \sin\theta - i\cos\theta}} \right)^n$$

$$= (\sin\theta + i\cos\theta)^n$$

$$= \left(\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)\right)^n$$

$$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right) = R.H.S$$