$x + \frac{1}{x} = 2\cos\theta$ prove that $x^n + \frac{1}{x^n} = 2\cos n\theta$

Sol. $x + \frac{1}{x} = 2\cos\theta$

$x^2 + 1 = 2x\cos\theta$

$x^2 - 2x\cos\theta + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2(1)}$$

$$= \frac{\cancel{2}\left(\cos\theta \pm \sqrt{(-1)(1-\cos^2\theta)}\right)}{\cancel{2}}$$

$$= \cos\theta \pm \sqrt{i^2 \cdot \sin^2\theta}$$

$$= \cos\theta \pm i\sin\theta.$$

$$x = (\cos\theta + i\sin\theta),\ (\cos\theta - i\sin\theta)$$

when $\quad x = \cos\theta + i\sin\theta$

$$x^n + \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + \frac{1}{(\cos\theta + i\sin\theta)^n}$$

$$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos n\theta + i\cancel{\sin n\theta} + \cos n\theta + i(-\cancel{\sin n\theta})$$

$$= 2\cos n\theta = R.H.S$$

when $x = \cos\theta - i\sin\theta$

$$x^n + \frac{1}{x^n} = (\cos\theta - i\sin\theta)^n + \frac{1}{(\cos\theta - i\sin\theta)^n}$$

$$= \cos n\theta - i\sin n\theta + (\cos\theta - i\sin\theta)^{-n}$$

$$= \cos n\theta - i\cancel{\sin n\theta} + \cos n\theta + i\cancel{\sin n\theta}$$

$$= 2\cos n\theta = R.H.S.$$

Hence $x^n + \dfrac{1}{x^n} = 2\cos n\theta.$