Trigonometry And Matrices : De Moivre's Theorem

If α and β be the roots of $x^2 - 2x + 4 = 0$, Prove that

(i) $\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$

(ii) $\alpha^6 + \beta^6 = 128$

Sol.

$$x^2 - 2x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm 2\sqrt{1-4}}{2}$$

$$= \frac{\cancel{2}(1 \pm \sqrt{-3})}{\cancel{2}} = 1 \pm i\sqrt{3}.$$

$\alpha = 1+i\sqrt{3}$ $\beta = 1-i\sqrt{3}.$

Let $1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$

By Comparing.

$$r\cos\theta = 1 \quad \text{—(I)}$$

$$r\sin\theta = \sqrt{3} \quad \text{—(II)}$$

By squaring and adding

$$r^2(\cos^2\theta + \sin^2\theta) = 1+3$$

$$r^2 = 4$$

$$\boxed{r = 2.}$$

Divide (II) by (I)

$$\frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{3}}{1}$$

$\tan\theta = \sqrt{3}.$

$\underline{\underline{\theta = 60^\circ}}$

(i) $\alpha^n + \beta^n = \left(r(\cos\theta + i\sin\theta)\right)^n + \left(r(\cos\theta - i\sin\theta)\right)^n$

$= r^n\left(\cos n\theta + i\,\cancel{\sin n\theta} + \cos n\theta - i\,\cancel{\sin n\theta}\right)$

$= r^n\left(2\cos n\theta\right)$

$= 2^n \cdot 2\cos n(60^\circ)$

$= 2^{n+1}\cos n\pi/3 \quad = \underline{\underline{R.H.S}}$

$$\alpha^6+\beta^6 = 2^{6+1}\cos\frac{6\pi}{3}$$

Put $n=6$

$$= 2^7\cdot\cos 2\pi = 128 = R.H.S$$