If $(1+x)^n = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \ldots\ldots$

show that

(i) $P_0 - P_2 + P_4 \ldots\ldots = 2^{n/2} \cos \frac{n\pi}{4}$

(ii) $P_1 - P_3 + P_5 - \ldots\ldots = 2^{n/2} \sin \frac{n\pi}{4}$

(iii) $P_0 + P_4 + P_8 + \ldots\ldots = 2^{n-2} + 2^{\frac{n-2}{2}} \cos \frac{n\pi}{4}$

Sol.

$$(1+x)^n = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + P_5 x^5 + \text{-----}$$

Let $x = i$

$$(1+i)^n = P_0 + P_1 i + P_2 i^2 + P_3 i^3 + P_4 i^4 + P_5 i^5 + \text{-----}$$

$$= P_0 + P_1 i + P_2(-1) + P_3(-i) + P_4(1) + P_5(i) + \text{-----}$$

$$(1+i)^n = (P_0 - P_2 + P_4 - \text{-----}) + i(P_1 - P_3 + P_5 - \text{-----}) \quad \text{--- (1)}$$

Let

$1 + i = r(\cos\theta + i\sin\theta)$

By Comparing real and imaginary parts

$r\cos\theta = 1$ — (ii)

$r\sin\theta = 1$ — (iii)

By squaring and adding (ii) & (iii)

$r^2(\cos^2\theta + \sin^2\theta) = 1 + 1$

$r^2 = 2$

$r = \sqrt{2}$

Now Divide III By II

$$\frac{r\sin\theta}{r\cos\theta} = 1$$

$$\tan\theta = 1$$

$$\theta = \pi/4$$

$$1+i = \sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)$$

$$(1+i)^n = \left[\sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)\right]^n$$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right] \quad \text{— IV}$$

$$(P_0 - P_2 + P_4 - \ldots\ldots) + i(P_1 - P_3 + P_5 - \ldots\ldots)$$

$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \left\{\text{from (I) and (IV)}\right\}$$

$$P_0 - P_2 + P_4 - \ldots\ldots = 2^{n/2}\cos n\pi/4$$

$$P_1 - P_3 + P_5 - \ldots\ldots = 2^{n/2}\sin\frac{n\pi}{4}$$

$$(1+x)^n = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + P_5 x^5 + \ldots$$

Put $x = 1$.

$$2^n = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + \ldots\ldots \quad \text{— (V)}$$

Put $x = -1$

$$0 = P_0 - P_1 + P_2 - P_3 + P_4 - P_5 + P_6 - P_7 + \cdots \quad \text{(VI)}$$

Add (V) and (VI)

$$2^n = 2[P_0 + P_2 + P_4 + P_6 + P_8 + \cdots]$$

$$2^{n-1} = P_0 + P_2 + P_4 + P_6 + P_8 + \cdots \quad \text{— (VII)}$$

$$P_0 - P_2 + P_4 - P_6 + \cdots = 2^{n/2} \cos \frac{n\pi}{4} \quad \text{— (VIII)}$$

Add (VII) and (VIII)

$$2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4} = 2[P_0 + P_4 + P_8 + \ldots]$$

$$2^{n-2} + 2^{n/2 - 1}\cos\frac{n\pi}{4} = P_0 + P_4 + P_8 + \ldots$$

$$P_0 + P_4 + P_8 + \ldots = 2^{n-2} + 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$$

Hence Proved