$\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$ and $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

Prove that

(i) $\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$

Sol. $a = \cos\alpha + i\sin\alpha$

$b = 2\cos\beta + i2\sin\beta$

$c = 3\cos\gamma + i3\sin\gamma$

$$a+b+c = (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma)$$

$$= 0 + i0$$

$$a+b+c = 0$$

$$a^3 + b^3 + c^3 = 3abc$$

$$(\cos\alpha + i\sin\alpha)^3 + (2\cos\beta + 2i\sin\beta)^3 + (3\cos\gamma + i3\sin\gamma)^3$$

$$= 3(\cos\alpha + i\sin\alpha)\,2(\cos\beta + i\sin\beta)\,3(\cos\gamma + i\sin\gamma)$$

$$\cos 3\alpha + i\sin 3\alpha + 2^3(\cos\beta + i\sin\beta)^3 + 27(\cos\gamma + i\sin\gamma)^3$$

$$= 18\ \text{cis}\,\alpha \cdot \text{cis}\,\beta \cdot \text{cis}\,\gamma$$

$$\cos 3\alpha + i\sin 3\alpha + 8\cos 3\beta + 8i\sin 3\beta + 27\cos 3\gamma + 27i\sin 3\gamma =$$

$$18\ \text{cis}(\alpha + \beta + \gamma)$$

$$\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma) =$$

$$18\{\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\}$$

$$\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$$

$$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$$

Hence Proved.