Solve the equation $x^5-1=0$ and show that the sum of n^{th} powers of the roots always vanishes unless n be a multiple of 5. n being an integer.

Sol

$$x^5-1=0$$

$$x^5=1$$

$$x=(1)^{1/5}$$

$$x=(\cos 0+i\sin 0)^{1/5}$$

$$=\left(\cos(2n\pi+0)+i\sin(2n\pi+0)\right)^{1/5}$$

$$=\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}$$

where $n=0,1,2,3,4$

$$(\cos 0+i\sin 0),\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right),\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)$$

$$\left(\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right),\left(\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}\right)$$

nth Power of Roots

$$(1)^n,\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^n,\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)^n,$$

$$\left(\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right)^{n},\left(\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}\right)^{n}$$

$$1,\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right),\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right)^{2}$$

$$\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right)^{3},\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right)^{4}$$

$$1,\ t,\ t^{2},\ t^{3},\ t^{4}$$

Case I   n is not multiple of 5.

$$1+t+t^2+t^3+t^4$$

$$\frac{1(1-t^5)}{1-t} = \frac{1-\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right)^5}{1-t}$$

$$= \frac{1-(\cos 2n\pi + i\sin 2n\pi)}{1-t}$$

$$= \frac{1-(1+0)}{1-t} = 0$$

Case II $\quad n$ is multiple of 5

Let $n = 5k$

$$1 + t + t^2 + t^3 + t^4$$

$$1 + \left(\cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}\right) + \left(\cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}\right)^2$$

$$+ \left(\cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}\right)^3 + \left(\cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5}\right)^4$$

$$= 1 + \left(\cos 2k\pi + i\sin 2k\pi\right) + \left(\cos(2k\pi) + i\sin 2k\pi\right)^2$$

$$+ (\cos 2k\pi + i\sin 2k\pi)^3 + (\cos 2k\pi + i\sin 2k\pi)^4$$

$$= 1 + (1+0) + (1+0)^2 + (1+0)^3 + (1+0)^4$$

$$= 1+1+1+1+1 = 5 \text{ Ans.}$$