Derivative of Inverse Hyperbolic functions : Example

$$y = e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} + \sinh^{-1}(\operatorname{sech} x)$$

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{— } (1)$$

$$u = e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)}$$

$$\frac{du}{dx} = e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} \frac{d}{dx}\left[\tanh^{-1}\left(\frac{2x}{1+x^2}\right)\right]$$

$$= e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} \left[\frac{1}{1-\left(\frac{2x}{1+x^2}\right)^2}\right] \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$= e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} \left[\frac{1}{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}\right] \left[\frac{(1+x^2)(2)-2x(2x)}{(1+x^2)^2}\right]$$

$$= e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} \left[\frac{\cancel{(1+x^2)^2}}{1+x^4+2x^2-4x^2}\right] \left[\frac{2+2x^2-4x^2}{\cancel{(1+x^2)^2}}\right]$$

$$= e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)}\left\{\frac{1}{x^4+1-2x^2}\cdot(2-2x^2)\right\}$$

$$= e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)}\left\{\frac{1}{(1-x^2)^{\cancel{2}}}\,2\cancel{(1-x^2)}\right\}$$

$$\frac{du}{dx} = e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)}\left(\frac{2}{(1-x^2)}\right) \quad \text{— } (1)$$

$$v = \sinh^{-1}(\operatorname{sech} x)$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1+\operatorname{sech}^2 x}}\,\frac{d}{dx}(\operatorname{sech} x)$$

$$= \frac{1}{\sqrt{1+\operatorname{sech}^2 x}} \cdot (-\operatorname{sech}x \tanh x) \quad \text{—(iii)}$$

Now from (i), (ii) and (iii)

$$\frac{dy}{dx} = e^{\tanh^{-1}\left(\frac{2x}{1+x^2}\right)} \cdot \frac{2}{(1-x^2)} + \frac{1}{\sqrt{1+\operatorname{sech}^2 x}}(-\operatorname{sech}x \tanh x)$$

Ans.