

Derivative of Inverse Hyperbolic functions : Example

Prove that $\tanh^{-1}x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$ $-1 < x < 1$ then also find the derivative.

Sol $y = \tanh^{-1}x$

$$\tanh y = x$$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{x}{1}$$

By Componendo and dividendo

$$\frac{e^y - \cancel{e^{-y}} + e^y + \cancel{e^{-y}}}{\cancel{e^y} - e^{-y} - \cancel{e^y} - e^{-y}} = \frac{x+1}{x-1}$$

$$\frac{\cancel{2}e^y}{-\cancel{2}e^{-y}} = \frac{x+1}{x-1}$$

$$+ e^y \cdot e^y = \frac{1+x}{1-x}$$

$$e^{2y} = \frac{1+x}{1-x}$$

Taking log Both sides

$$\log e^{2y} = \log \left(\frac{1+x}{1-x} \right)$$

$$2y = \log \left(\frac{1+x}{1-x} \right)$$

$$y = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$\log \left(\frac{1+x}{1-x} \right)$ is defined only when $\frac{1+x}{1-x} > 0$

Case I

$$1-x > 0$$

$$1 > x$$

$$\boxed{x < 1}$$

$$\text{Now } \frac{1+x}{1-x} > 0 \Rightarrow 1+x > 0$$

$$x > -1$$

$$-1 < x < 1$$

Case II

$$1 - x < 0$$

$$x > 1$$

$$\frac{1+x}{1-x} > 0$$

> 0

\Rightarrow

$$1+x < 0$$

$$x < -1$$

which is not possible



$$\Rightarrow \frac{1+x}{1-x} > 0 \quad \text{When} \quad \frac{1-x}{-1 < x < 1}$$

$$\tanh^{-1} x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right) \quad -1 < x < 1$$

$$y = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$$y = \frac{1}{2} [\log(1+x) - \log(1-x)]$$

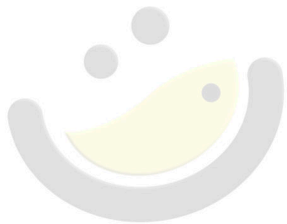
$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{1+x} \cdot 1 - \frac{1}{(1-x)} (-1) \right]$$

$$= \frac{1}{2} \left[\frac{1-x + 1+x}{(1-x)(1+x)} \right]$$

$$= \frac{1}{2} \left[\frac{2}{1-x^2} \right] = \frac{1}{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2} \quad \underline{\text{Ans.}}$$



OMG { MATHS }
The poetry of logical ideas.