Derivative of Inverse Hyperbolic functions : Example

Prove that $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ $-1<x<1$ then also find the derivative.

Sol: $y = \tanh^{-1}x$

$\tanh y = x$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{x}{1}$$

By Componendo and dividendo

$$\frac{e^y - \cancel{e^{-y}} + e^y + \cancel{e^{-y}}}{\cancel{e^y} - e^{-y} - \cancel{e^y} - e^{-y}} = \frac{x+1}{x-1}$$

$$\frac{\cancel{2} e^y}{-\cancel{2} e^{-y}} = \frac{x+1}{x-1}$$

$$+e^y \cdot e^y = \frac{1+x}{+(1-x)}$$

$$e^{2y} = \frac{1+x}{1-x}$$

Taking log Both sides

$$\log e^{2y} = \log\left(\frac{1+x}{1-x}\right)$$

$$2y = \log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$\log\left(\frac{1+x}{1-x}\right)$ is defined only when $\frac{1+x}{1-x} > 0$

Case I $\quad 1-x > 0 \qquad 1 > x \qquad \boxed{x<1}$

Now $\frac{1+x}{1-x} > 0 \Rightarrow 1+x > 0$

$x > -1$

$$\boxed{-1 < x < 1}$$

Case II

$1 - x < 0$

$$\boxed{x > 1}$$

$$\frac{1+x}{1-x} > 0 \Rightarrow 1 + x < 0$$

$x < -1$

Which is not Possible

2) $\dfrac{1+x}{1-x} > 0$ when $1-x > 0$

$-1 < x < 1$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+x}\cdot 1 - \frac{1}{(1-x)}(-1)\right]$$

$$= \frac{1}{2}\left[\frac{1-\cancel{x}+1+\cancel{x}}{(1-x)(1+x)}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-x^2}\right] = \frac{1}{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$$ Ans.