Limit and Continuity

Examine the Continuity of the function at $x = 0$

$$f(x) = \begin{cases} \dfrac{\sqrt{1-\cos 2x}}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Sol:

$$f(x) = \frac{\sqrt{1-\cos 2x}}{x} \qquad x \neq 0$$

$$= \frac{\sqrt{2\sin^2 x}}{x} \qquad x \neq 0$$

$$f(x) = \frac{\sqrt{2}\,|\sin x|}{x} \qquad x \neq 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sqrt{2}\,|\sin x|}{x}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\,|\sin(0-h)|}{(0-h)} \qquad \left|\; \begin{array}{l} x \to 0^- \\ h \to 0 \end{array} \right.$$

$$= \frac{\sqrt{2}\,\lim_{h \to 0} |\sin(-h)|}{-h}$$

$$= -\sqrt{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$\left\{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right.$$

$$= -\sqrt{2}(1) = \underline{-\sqrt{2}} \quad \text{—(I)}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\sqrt{2}\frac{|\sin x|}{x}$$

$$\left|\begin{array}{l} x\to 0^+ \\ h\to 0 \end{array}\right.$$

$$\underline{x = 0+h}$$

$$= \lim_{h\to 0}\sqrt{2}\frac{|\sin(0+h)|}{(0+h)}$$

$$= \sqrt{2}\lim_{h\to 0}\frac{\sin h}{h} = \sqrt{2}\cdot 1 = \sqrt{2} \quad \text{—(II)}$$

from (i) & (ii)

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Discontinous

Discontinuity of first kind.