Limit and Continuity

Evaluate

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \infty$$

So: By Binomial expansion

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \dots \infty$$

$$= 1 + 1 + \frac{\cancel{n}\cdot\cancel{n}\left(1-\frac{1}{n}\right)}{2!}\cdot\frac{1}{\cancel{n^2}} + \frac{\cancel{n^3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}\cdot\frac{1}{\cancel{n^3}}$$

$$+ - - - \infty$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \dots \infty$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left[1 + 1 + \frac{(1-1/n)}{2!} + \frac{(1-1/n)(1-2/n)}{3!} + \dots \infty\right]$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty$$

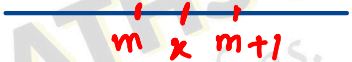
$$= e.$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \quad \forall\, n \in \mathbb{N}.$$

Let x be any +ve real No.

By Archimedean property

$$m \leq x < m+1$$



$$\frac{1}{m} \geq \frac{1}{x} > \frac{1}{m+1}$$

$$1+\frac{1}{m} \geq 1+\frac{1}{x} > 1+\frac{1}{m+1}$$

Now

$x \to \infty$, $m \to \infty$ $m+1 \to \infty$

$$\left(1+\frac{1}{m}\right)^{m+1} \geq \left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{m+1}\right)^{m}$$

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m+1} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\left(1+\frac{1}{m}\right)$$

$$= e\cdot 1 = e.$$

$$\lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)^{m} = \lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)^{m+1}\left(1+\frac{1}{m+1}\right)^{-1}$$

$$= e\cdot 1 = e$$

By Squeeze Principle

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x} = e.$$