Calculus

Uniqueness of Limit

Thm If $\lim_{x \to a} f(x)$ exists finitely, then prove that it is unique.

Proof Let $\lim_{x \to a} f(x) = l$

and $\lim_{x \to a} f(x) = l'$ where $l' \neq l$

Also let $\epsilon > 0$

$$\epsilon = \frac{l' - l}{2} > 0$$

Now $\lim_{x \to a} f(x) = l$. for $\epsilon > 0$

we will find a no. $\delta_1 > 0$ s.t.

$$|f(x) - l| < \epsilon \quad \forall \quad 0 < |x-a| < \delta_1 \quad \text{— (I)}$$

also $\lim_{x \to a} f(x) = l'$ for $\epsilon > 0$

we will find a No. $\delta_2 > 0$ s.t.

$$|f(x) - l'| < \epsilon \quad \forall \quad 0 < |x-a| < \delta_2 \quad \text{— (II)}$$

$\delta = \text{Min}\{\delta_1, \delta_2\}$

from (i) & (ii)

$|f(x) - l| < \epsilon \quad \forall \quad 0 < |x-a| < \delta$ — (iii)

$|f(x) - l'| < \epsilon \quad \forall \quad 0 < |x-a| < \delta$ — (iv)

Now $\quad l' - l = |l' - l| \qquad l' > l.$

$= |l' - l + f(x) - f(x)|$

$= |\{f(x) - l\} + \{l' - f(x)\}|$

$\leq |f(x) - l| + |l' - f(x)|$

$|a+b| \leq |a| + |b|$

$= |f(x) - l| + |f(x) - l'|$ $\quad [\because |x| = |-x|]$

$< \epsilon + \epsilon \quad \forall\, 0 < |x-a| < \delta$ [from (iii) & (iv)]