Limit and Continuity

Squeeze Principle

If $\underbrace{f(x)}_{l} < \underbrace{g(x)}_{?} < \underbrace{h(x)}_{l}$ for all x in some deleted neighbourhood of a and

$\lim\limits_{x \to a} f(x) = l = \lim\limits_{x \to a} h(x)$ then $\lim\limits_{x \to a} g(x) = l.$

Proof

$\lim\limits_{x \to a} f(x) = l$ (given)

for $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$|f(x) - l| < \epsilon$ for $0 < |x-a| < \delta$, [By def.]

$l-\epsilon < f(x) < l+\epsilon$ for $0<|x-a|<\delta_1$ —(i)

$\lim_{x\to a} h(x) = l$

for $\epsilon>0$ $\exists\, \delta>0$ s.t

$|h(x)-l| < \epsilon$ for $0<|x-a|<\delta_1$ [By def.]

$l-\epsilon < h(x) < l+\epsilon$ for $0<|x-a|<\delta_2$ —(ii)

Let $\delta = \text{Min}\{\delta_1, \delta_2\}$

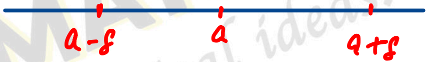
from (i) and (ii)

$l-\epsilon < f(x) < l+\epsilon$ for $0<|x-a|<\delta$ —(iii)

$l-\epsilon < h(x) < l+\epsilon$ for $0<|x-a|<\delta$ —(iv)

$f(x) < g(x) < h(x)$ for all x some deleted Nbd of a

$$\Rightarrow \quad f(x) < g(x) < h(x) \text{ for } |x-a| < \delta \quad \text{— (V)}$$



from (III), (IV) and (V)

$$l - \epsilon < f(x) < g(x) < h(x) < l + \epsilon \text{ for } 0 < |x-a| < \delta$$

$$l - \epsilon < g(x) < l + \epsilon \text{ for } 0 < |x-a| < \delta$$

$$-\epsilon < g(x) - l < \epsilon \quad \text{for } 0 < |x-a| < \delta$$

$$|g(x) - l| < \epsilon \quad \text{for } 0 < |x-a| < \delta$$

By def. of Limit

$$\lim_{x \to a} g(x) = l.$$

Hence proved.