Rabbe's Test for Convergence

3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$$

Then (*i*) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(*ii*) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(*iii*) Test fails if $l = 1$

4. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

Then (*i*) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(*ii*) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

(*iii*) Test fails if $l = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$u_n = \frac{1}{n(n+1)}$$

$$u_{n+1} = \frac{1}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \mathop{\mathrm{Lt}}_{n\to\infty} \frac{1}{\cancel{(n+1)}(n+2)} \times \frac{n\cancel{(n+1)}}{1}$$

$$= \lim_{n\to\infty} \frac{n}{n+2} = \mathop{\mathrm{Lt}}_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+2/n)} = 1.$$

$\Rightarrow$ Ratio test fails to check the Convergence

Now $\displaystyle \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)$

$$= \lim_{n\to\infty} n\left[\frac{1}{n(n\cancel{+1})} \times \frac{(n\cancel{+1})(n+2)}{1} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{n}+2-\cancel{n}}{n}\right] = \lim_{n\to\infty} n\left[\frac{2}{n}\right] = 2 > 1$$

Hence By Rabbe's test

$$\sum_{n=1}^{\infty} u_n \text{ is Cgt.}$$

(2) $$\sum_{n=1}^{\infty} u_n = \frac{2\cdot 4\cdot 6 \cdots\cdots 2n}{1\cdot 3\cdot 5 \cdots\cdots (2n+1)}$$

$$u_n = \frac{2\cdot 4\cdot 6 \cdots\cdots 2n}{1\cdot 3\cdot 5 \cdots\cdots (2n+1)}$$

$$u_{n+1} = \frac{2\cdot 4\cdot 6 \cdots\cdots 2n(2n+2)}{1\cdot 3\cdot 5 \cdots\cdots (2n+1)(2n+3)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

$$\lim_{n\to\infty} = \frac{2.4.6\ \_\ \_\ \_\ \_\ 2n(2n+2)}{1.3.5\ \_\ \_\ \_\ \_\ (2n+1)(2n+3)} \times \frac{1.3.5\ \_\ \_\ \_\ \_\ (2n+1)}{2.4.6\ \_\ \_\ \_\ \_\ (2n)}$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+3} = \lim_{n\to\infty} \frac{\cancel{n}(2+2/n)}{\cancel{n}(2+3/n)} = 1.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

So Ratio Test fails to check the Convergence

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{2.4.6\,----\,2n}{1.3.5\,--\,(2n+1)} \times \frac{1.3.5\,--\,(2n+1)\,(2n+3)}{2.4.6\,--\,(2n)\,(2n+2)} - 1\right]$$

$$\lim_{n\to\infty} n\left[\frac{2n+3}{2n+2} - 1\right] = \lim_{n\to\infty} n\left[\frac{2\cancel{n}+3 - 2\cancel{n}-2}{2n+2}\right]$$

$$= \lim_{n\to\infty} \cancel{n}\left[\frac{1}{\cancel{n}(2+2/n)}\right] = \frac{1}{2} < 1.$$

$\therefore$ By Rabbe's test

$\sum_{n=1}^{\infty} u_n$ is Divergent.

(3) $\sum_{n=1}^{\infty} u_n = \dfrac{3.6.9 \ldots \ldots 3n}{7.10.13 \ldots \ldots (3n+4)} x^n$

$$u_n = \frac{3.6.9 \ldots \ldots 3n}{7.10.13 \ldots \ldots (3n+4)} x^n$$

$$u_{n+1} = \frac{3.6.9 \ldots \ldots 3n(3n+3)}{7.10.13 \ldots \ldots (3n+4)(3n+7)} x^{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{3.6.9 \ldots \ldots 3n(3n+3)}{7.10.13 \ldots \ldots (3n+4)(3n+7)} x^{n+1} \times \frac{7.10.13 \ldots (3n+4)}{3.6.9 \ldots (3n) x^n}$$

$$= \lim_{n\to\infty} \frac{3n+3}{3n+7}\, x$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\{3+3/n\}}{\cancel{n}\{3+7/n\}} \cdot x$$

$$= x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x.$$

By D' Alembert's Ratio test.

$\sum_{n=1}^{\infty} u_n$ is cgt when $x < 1$

dgt when $x > 1$.

test fails for $x = 1$.

for $\underline{\underline{x=1}}$ $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right]$

$$= \lim_{n\to\infty} n\left[\frac{3\cdot 6\cdot 9 \ldots 3n}{7\cdot 10\cdot 13 \ldots (3n+4)} \times \frac{7\cdot 10\cdot 13 \ldots (3n+4)(3n+7)}{3\cdot 6\cdot 9 \ldots 3n(3n+3)} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{3n+7}{3n+3} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4}{3n+3}\right] = \lim_{n\to\infty} \cancel{n}\left[\frac{4}{\cancel{n}(3+3/n)}\right]$$

$$= \frac{4}{3} > 1$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] > 1$$

By Rabbe's test

$\sum_{n=1}^{\infty} u_n$ is Cgt for $l > 1$. and $x = 1$.

Hence $\sum_{n=1}^{\infty} u_n$ is Cgt for $x \leq 1$

dgt for $x > 1$.