Limit and Continuity

Prove that

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a.$$

Proof: Let $a^x - 1 = y$. so that $y \to 0$ as $x \to 0$

$a^x = 1 + y$

Taking log both sides

$\log a^x = \log(1+y)$

$x \log a = \log(1+y)$

$$x = \frac{\log(1+y)}{\log a}$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\frac{\log(1+y)}{\log a}}$$

$$\lim_{y \to 0} \frac{y \log a}{\log(1+y)} = \log a \lim_{y \to 0} \frac{y}{\log(1+y)}$$

$$= \log a \lim_{y \to 0} \frac{1}{\frac{1}{y} \log(1+y)}$$

$$= \log a \lim_{y \to 0} \frac{1}{\log(1+y)^{1/y}}$$

$$= \log a \cdot \frac{1}{\log e} \qquad \left[\lim_{x \to 0} (1+x)^{1/x} = e\right]$$

$$\left[\because \log e = 1\right]$$

$$= \log a$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$

Hence Proved.