Limit and Continuity

Prove that

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

$$(1+x)^{1/x} = 1 + \frac{1}{x} \cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!} x^2 + \frac{\left(\frac{1}{x}\right)\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!} x^3$$

$$+ \text{-----}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \text{---}$$

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$$= 1 + 1 + \frac{\frac{1}{x}\left(\frac{1-x}{x}\right)}{2!}\cdot x^2 + \frac{\frac{1}{x}\left(\frac{1-x}{x}\right)\left(\frac{1-2x}{x}\right)}{3!}\cdot x^3 + \cdots$$

$$= 1 + 1 + \frac{1(1-x)}{\cancel{x^2}\cdot 2!}\cdot \cancel{x^2} + \frac{1(1-x)(1-2x)}{\cancel{x^3}\cdot 3!}\cdot \cancel{x^3} + \cdots$$

$$= 1 + \frac{1}{1!} + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \cdots$$

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}\left[1 + \frac{1}{1!} + \frac{(1-x)}{2!} + \frac{(1-x)(1-2x)}{3!} + \cdots\right]$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^1 = e$$

Hence

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

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